Cycle double cover conjecture

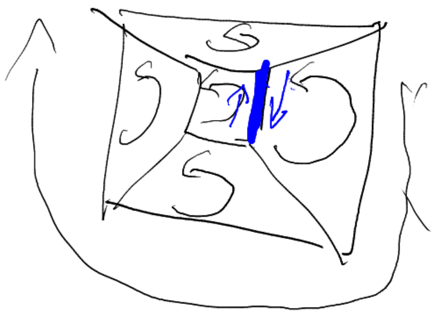
Conjecture 1. *Let G be a bridgeless graph. Then there is a collection of cycles such that every edge is covered by exactly two of them.*

- equivalently: "a collection of circles"
- stronger conjecture (k-**CDC**): "a collection of at most k cycles"
  Conjectured for $k = 5$, unknown for any $k \geq 5$.
- equivalently: "a collection of circles that can be k colored" OCDC
- stronger conjecture (oriented CDC): "a collection of oriented cycles such that every edge is covered once in each direction"
- even orientable 5-CDC can be true

cyklus = kr. def: sj. kružnice
cycle
circle

"cycle$_1$" = sj. kružnice barvy 1

cycle$_k$ = — " — barvy k

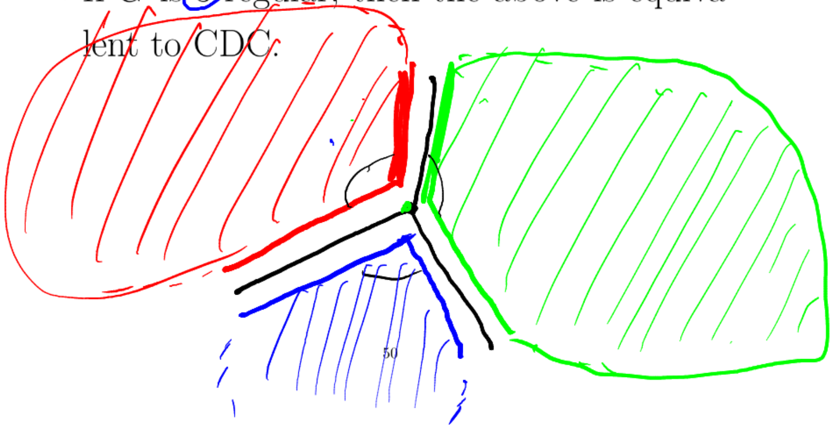


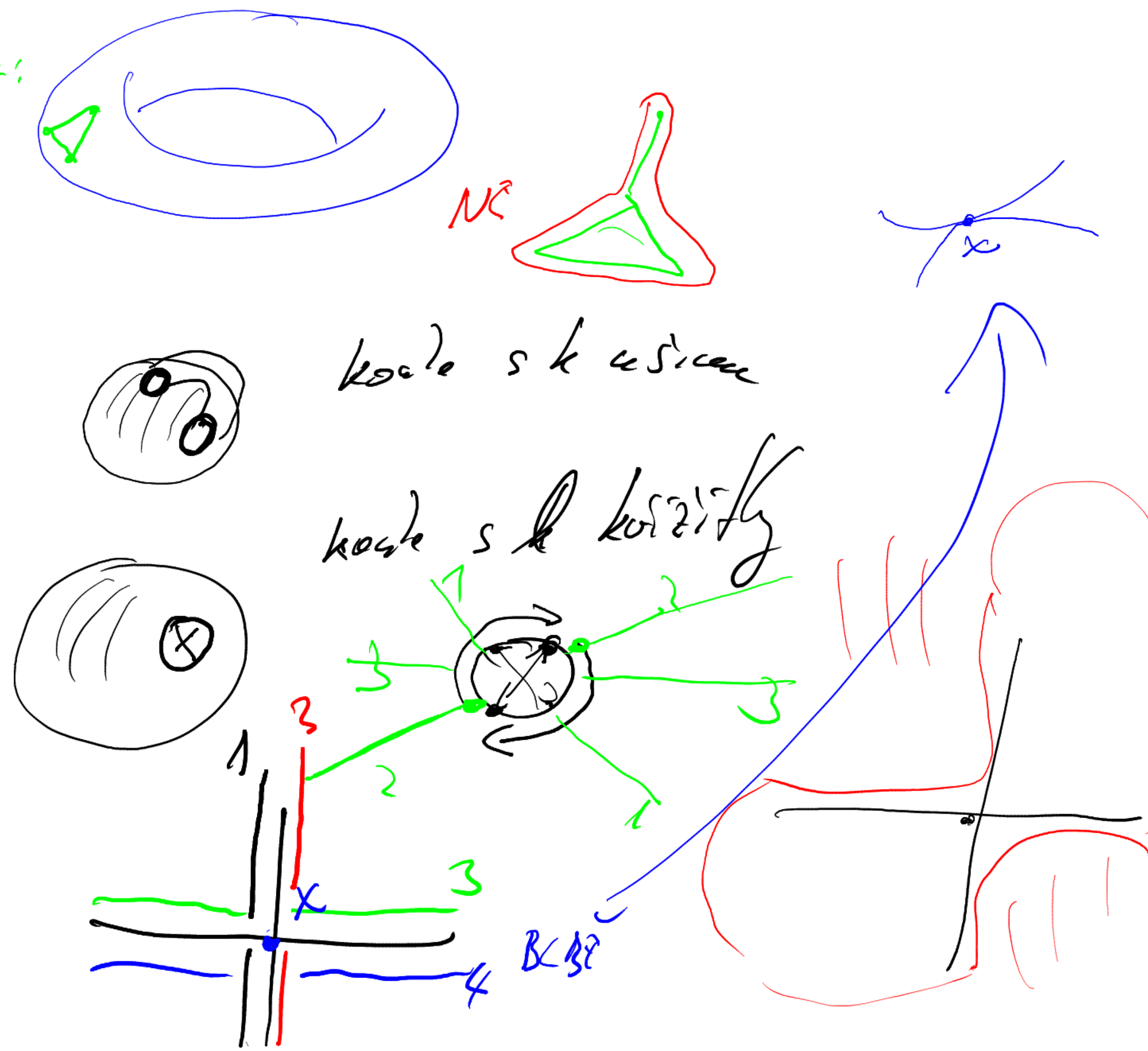
Relation with embedding

- 2-cell embedding of a graph: every face is homeomorphic to a disk
- circular 2-cell embedding: moreover, the boundary of each face is a circle
- 2-cell embedding on any surface implies CDC
- 2-cell embedding on an *orientable* surface implies OCDC

Conjecture 2. *Let G be a connected bridgeless graph. Then G has a circular 2-cell embedding on some surface.*

- If G is 3-regular, then the above is equivalent to CDC.





Theorem 27. *The following is equivalent for a graph G.*

1. *G has a 4-NZF.*
2. *G has has a 3-CDC.*
3. *G has has a 4-CDC.*
4. *G has has a 4-OCDC.*

Proof. Sketch: 1 and 2 are easily equivalent. $3 \Rightarrow 2$: Given a CDC C_1, C_2, C_3, C_4, the collection $C_1 \Delta C_2$, $C_1 \Delta C_3$, and $C_1 \Delta C_4$ is also a CDC. $2 \Rightarrow 4$: Given a CDC C_1, C_2, C_3, define a flow f_i with values ± 1 along C_i and 0 elsewhere. It is easy to check that $(f_1 + f_2 + f_3)/2$, $(f_1 - f_2 - f_3)/2$, $(-f_1 + f_2 - f_3)/2$, $(-f_1 - f_2 + f_3)/2$ are a 4-OCDC. □

$\boxed{1 \Leftrightarrow 2}$ 1: $\exists \varphi : E(G) \to \mathbb{Z}_2^2 \setminus \{(0,0)\}$ ~ tok

$1 \Rightarrow 2$

$\varphi = (\varphi_1, \varphi_2)$, $C_i := \operatorname{supp} \varphi_i$

$C_1 \cup C_2 = E(G)$ C_1, C_2 cykly

$C_3 = C_1 \Delta C_2$ ~ cyklus

(C_1, C_2, C_3) je 3-CDC

$2 \Rightarrow 1$ $(C_1, C_2) \rightsquigarrow \varphi_1, \varphi_2 \rightsquigarrow \varphi$

$\boxed{3 \Rightarrow 2}$ $e \in C_1 \cap C_2 \to e \in C_1 \Delta C_3,\ C_1 \Delta C_4$

$e \in C_2 \cap C_3 \to e \in C_1 \Delta C_2,\ C_1 \Delta C_4$

$\boxed{2 \Rightarrow 4}$ $\forall e$ $f_1(e), f_2(e), f_3(e) \in \{\pm 1, 0\}$

jedna 0!

hodnota $\in \{0, \pm 1\}$

g_1, g_2, g_3, g_4

$\sum g_i(e) = 0$

$f_1(e) = +1,\ f_2(e) = -1,\ f_3(e) = 0$

$g_1(e) = 0,\ g_2(e) = 1,\ g_3(e) = -1$

$g_4 = 0$

Theorem 28. *Every graph with k-OCDC has a k-NZF. The opposite implication is known only for $k \leq 4$: Every graph with a k-NZF ($k \leq 4$) has a k-OCDC.* $\boxed{k}$

Proof. Given a k-OCDC, find a flow f_i along the cycle C_i. The flow $\sum_{i=1}^{n} i f_i$ is a k-NZF. The other implication is easy for $k = 2$ and already proved for $k = 4$. The remaining case $k = 3$ relies on Exercise 29. □

Exercise 29. *Let f be a flow on a digraph G such that $0 < f(e) \leq k$ for every edge of G. Prove that there are $\{0, 1\}$-valued flows f_1, …, f_k such that $f = \sum_{i=1}^{k} f_i$.*

$f_i = \pm 1$ na C_i

$= 0$ jinde

k

$\sum i f_i$

tok

hodnota: $a - b$

$a, b \in \{1, \dots, k\}$

$a \neq b$

…… $\in \{-(k-1), \dots, -1,$

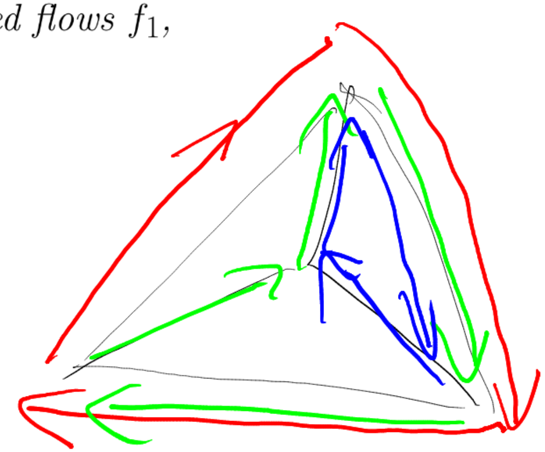
$1, \dots, k-1\}$

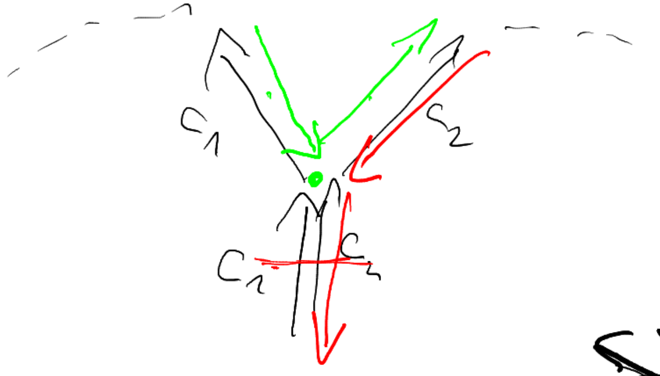
3-NZF

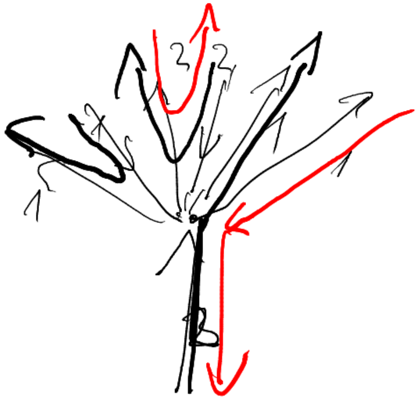
$\exists$ orientace $\vec{G}$ f_i tž. $f(e) \in \{1, 2\}$

$f = f_1 + f_2$ $\quad f_1, f_2$ … C_1, C_2

$C_1, C_2, \; C_1 \Delta C_2$ je 3-CDC ? orientace ?







$f_1, f_2 \in \{0, \pm 1\}$

$\underbrace{f_1, -f_2, f_2 - f_1}_{\Sigma = 0} \in \{0, \pm 1\}$

tj. je splněna rce $\Sigma = 0$

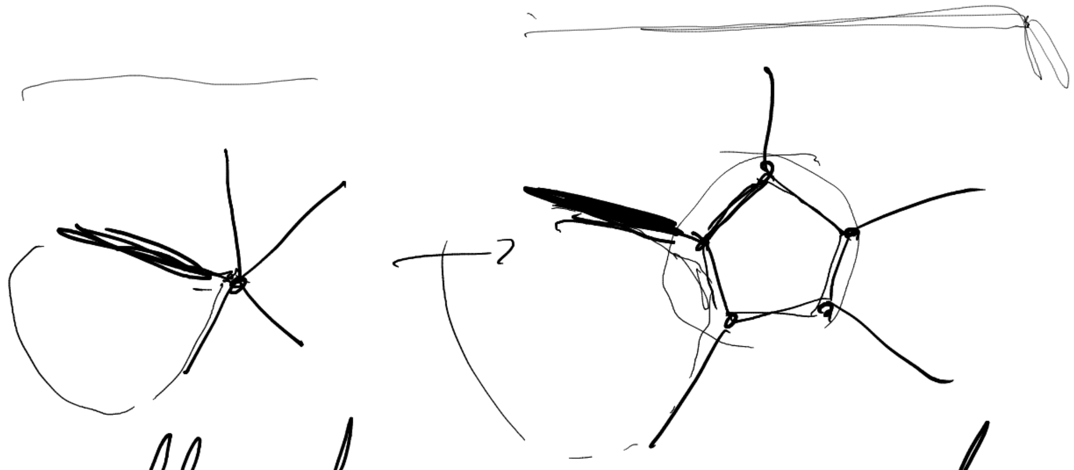
$C_3 = C_1 \triangle C_2$

orient. opačná než C_1, C_2

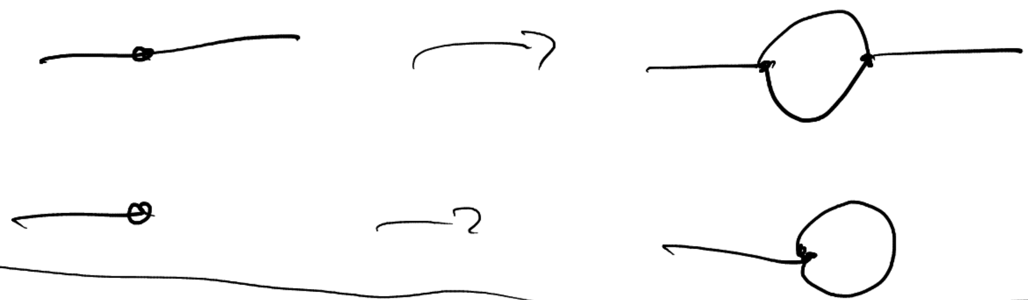
ověřit: je to konzistentní orientace

Theorem 30. *Suppose every cubic bridgeless graph has a CDC. Then every bridgeless graph has a CDC as well.*

Proof. Let G be a bridgeless graph. We let $H(G)$ denote the graph obtained from G by replacing a vertex v by a circuit of length $\deg(v)$ (we allow $\deg(v) \leq 2$ as well); we let $C(v)$ be the circuit corresponding to v. If uv is an edge in G, then we add any edge between $C(u)$ and $C(v)$ in $H(G)$, choosing the order arbitrarily. It is easy to verify that $H(G)$ is bridgeless. Further, any CDC in $H(G)$ yields easily a CDC in G by contracting each of the new circuits $C(v)$. □

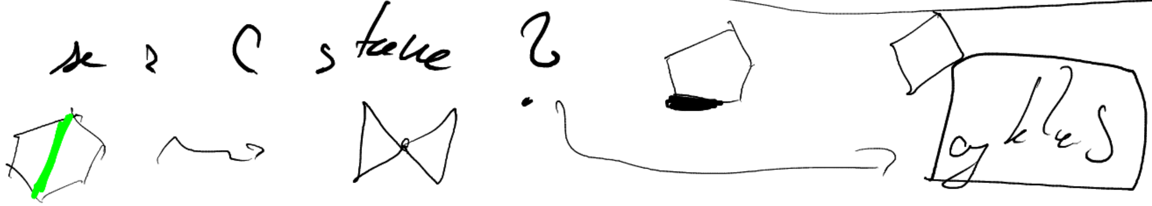


z lib. grafu → 3-reg. graf



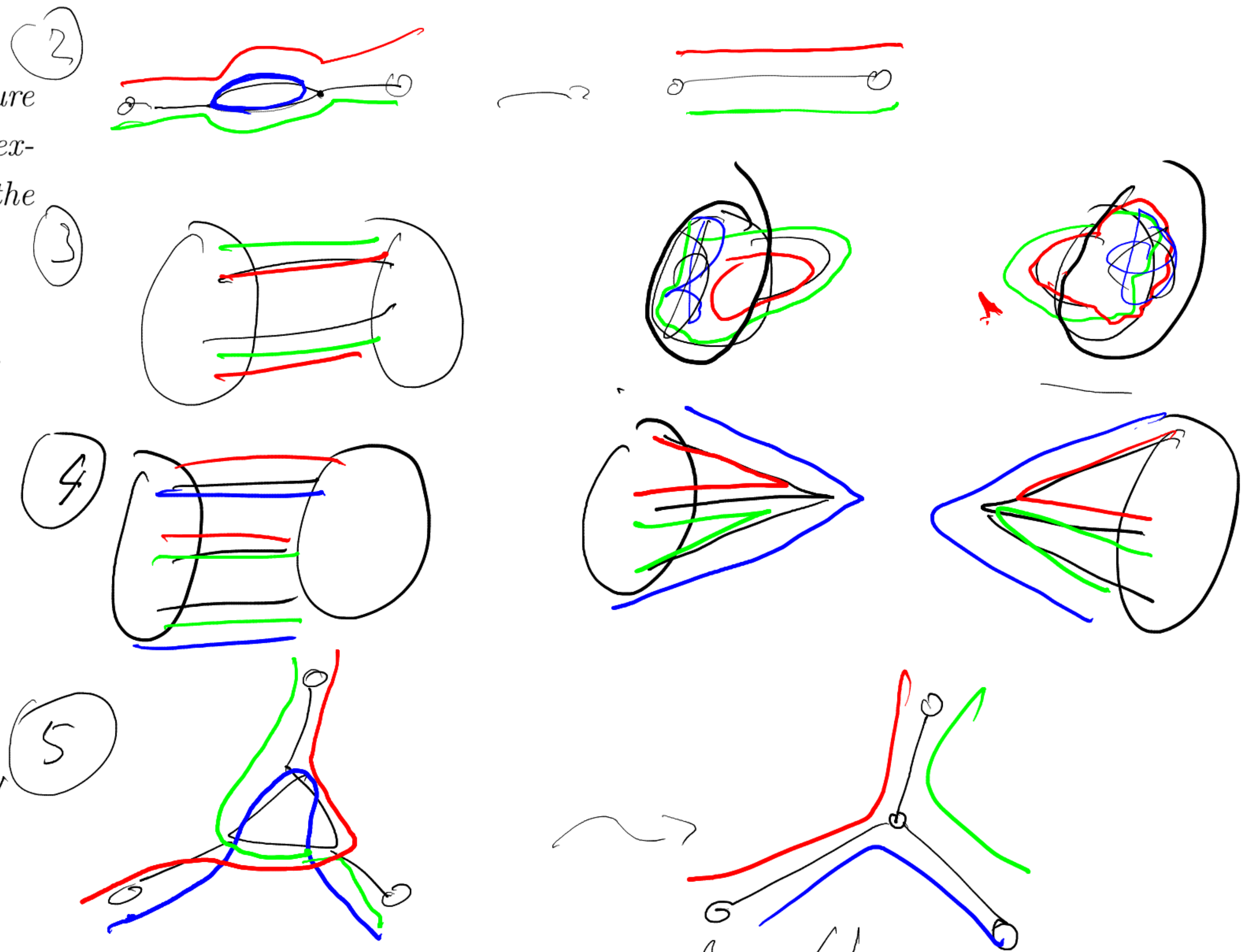
$G \supseteq C$ kružnice

$e \in E(G)$ ---> v G/e se z C stane ?



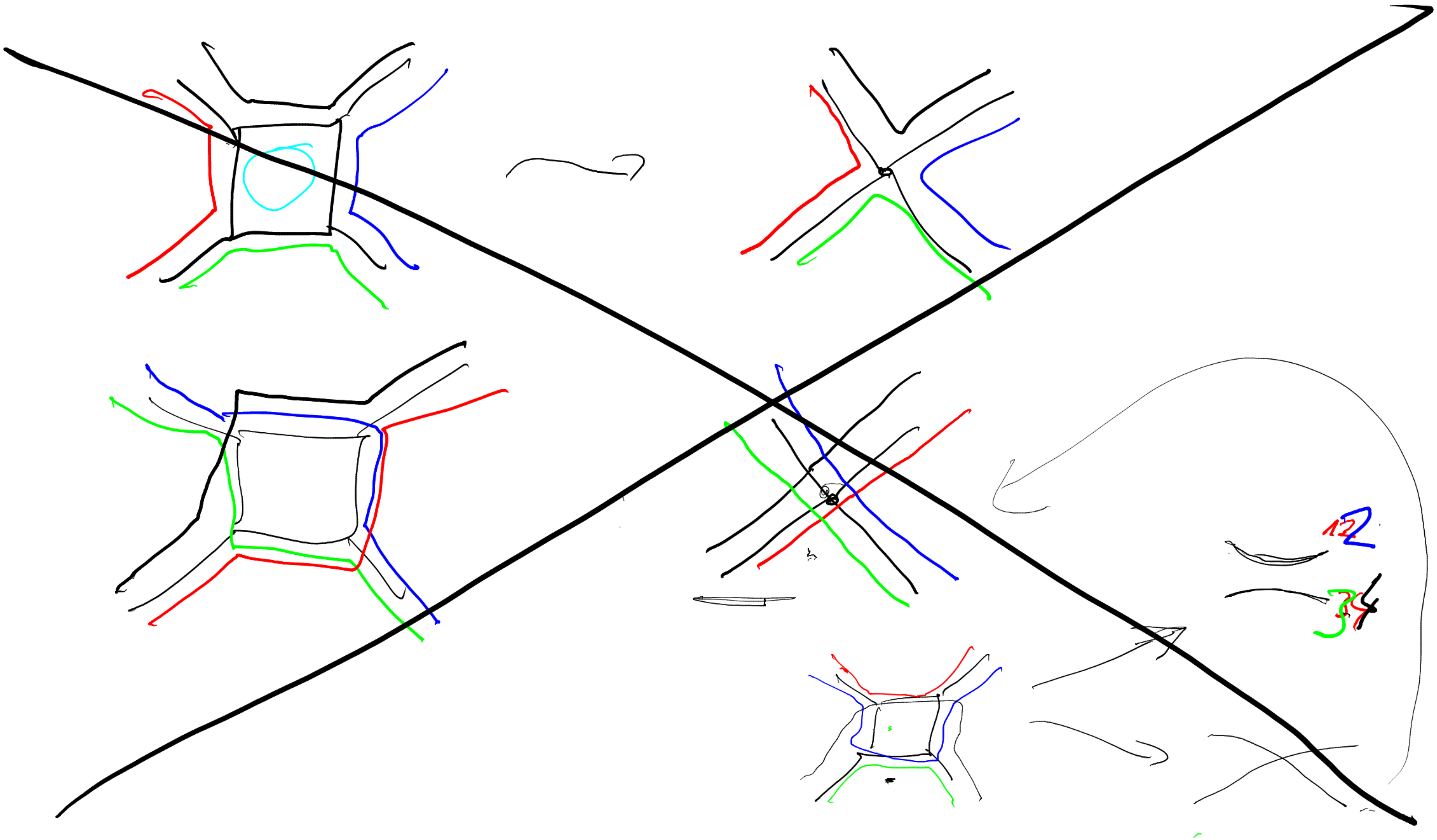
Theorem 31. *Suppose the CDC conjecture is false and let G be a minimal counterexample to it (i.e., a counterexample with the minimal number of edges). Then*

1. *G is cubic;*
2. *G does not contain two parallel edges;*
3. *G does not contain a 2-edge-cut;*
4. *G does not contain a 3-edge-cut;*
5. *G does not contain a C_3;*
6. *G does not contain a C_4;*
7. *G does not contain a C_{11};*
8. *G is not planar;*
9. *G does not contain a Hamilton cycle;*
10. *G does not contain a Hamilton path;*
11. *G has oddness at least 6;*
12. *G is not 3-edge-colorable.*

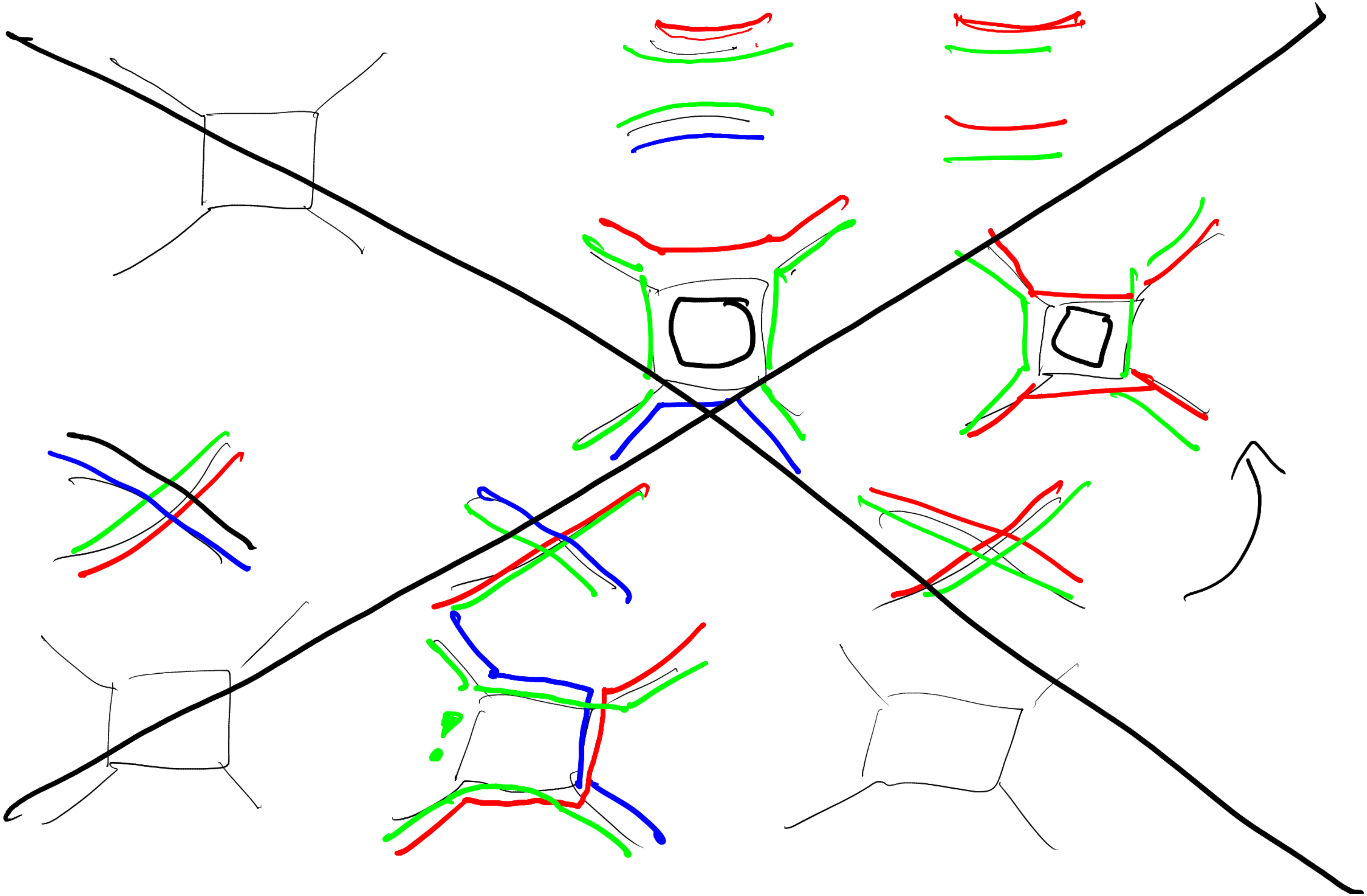


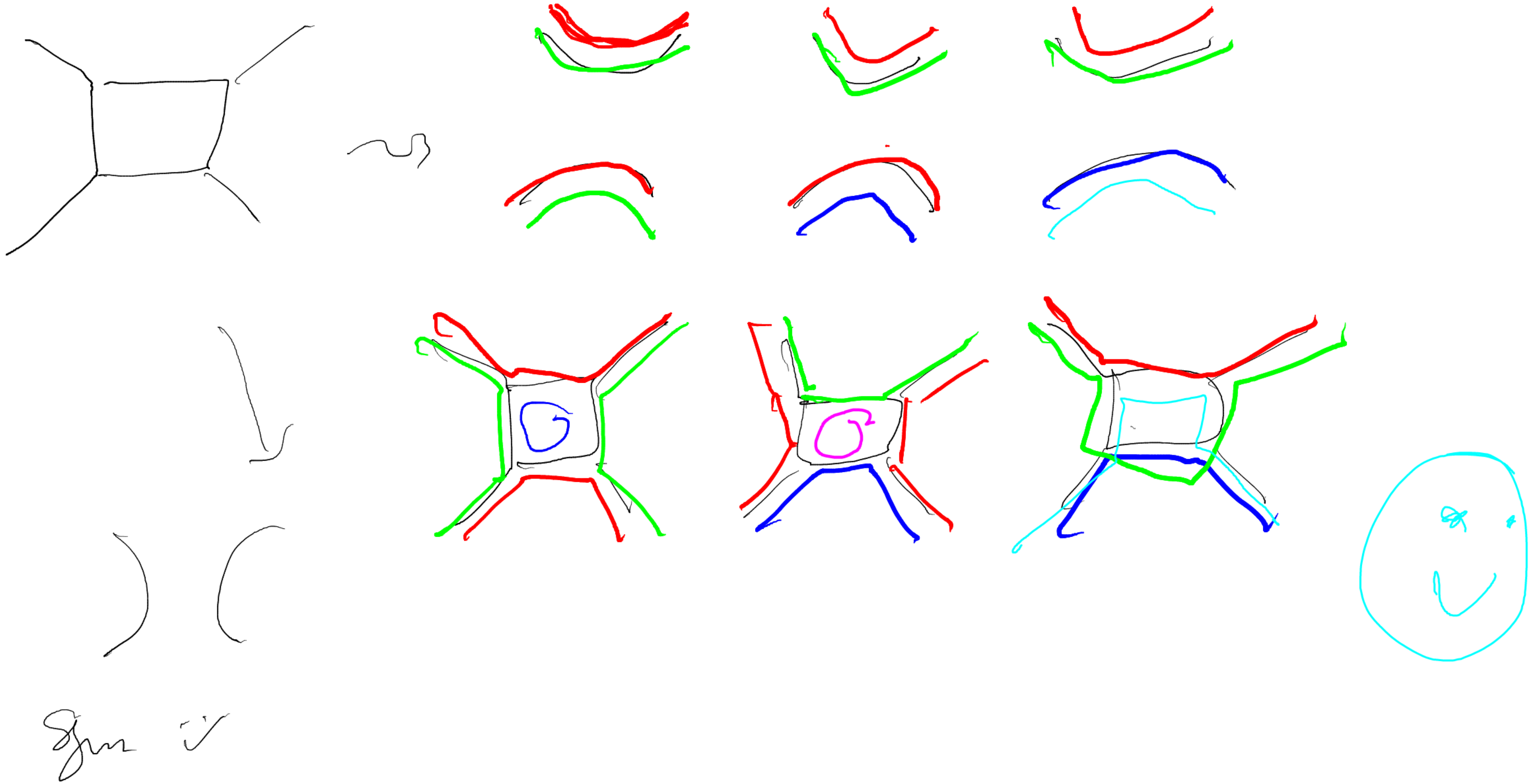
G kubický & C je HK v G ⟶ 3-obarvitelný

⟹ C je sudá ⟹ hr. 2-obarvitelná + E(G) − C je párování



12
34





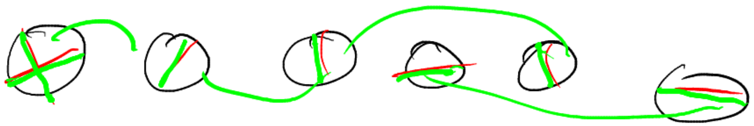
Proof. We will prove 1 and 6 later. 2, 3, 4, 5 are easy. 7 is the best-known in this direction so far (a computer search by Huck, 2000). 8 is easy: for a planar graph we may take the face-boundaries. We postpone 9 and 10 and ??11?? to Section **??**.

For 1: we apply the splitting lemma (and suppress vertices of degree 2) until we are left with a cubic graph. For 6: contract the 4-cycle and split, deal with the two cases. □

oddness grafu G min. k tž.
∃F ... 2-faktor G
tž. F má k lichých
kružnic

Věta G 3-regulární ⇒ ∃M ... p.p. v G
bez mostů

F = G − M je 2-regulární — sjednocení
kružnic



Splitting lemma

Lemma 32 (Splitting lemma, Fleischner/Mader)**.** *Let G be a connected bridgeless graph, v a vertex with $\deg v \geq 4$, and e_0, e_1, e_2 three of its incident edges. Suppose that $G_{[v:e_0,e_1,e_2]}$ is connected. (This in particular holds whenever G is 2-connected.) Then at least one of $G_{[v:e_0,e_1]}$, $G_{[v:e_0,e_2]}$, is bridgeless connected.*

Proof. Let the edge e_i connect v with v_i. Let $G' = G - \{e_0, e_1, e_2\}$ and consider decomposition of G' into edge 2-connected blocks. Next contract each block to a vertex, what we get is a forest, say, F. Let u and u_i be the vertices of F corresponding to v and v_i $(i = 0, 1, 2)$. As G is connected and bridgeless, the same is true for $F + \{uu_i : i = 0, 1, 2\}$. (In particular, the only leaves of F are among u, u_0, u_1, u_2.)

Also splitting, say, e_0, e_1 away from v corresponds to adding $F + \{uu_2, u_0u_1\}$ – for such graphs we need to check edge 2-connectivity. We have just two possibilities:

F **is disconnected.** As $G_{[v:e_0,e_1,e_2]}$ is connected and G bridgeless, the component containing u contains also (exactly) one u_i. Moreover, this component is a path connecting u with u_i. The other important vertices (say u_j, u_k, where $\{i, j, k\} = \{0, 1, 2\}$) are in the other component, this component is a $u_j - u_k$ path. In this case, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity. Easily, on of these includes the desired cases (as $0 \in \{i, j, k\}$). See the first case in Figure **??**.

F **is connected.** Let T be the minimal subtree containing u_0, u_1, u_2. Let $w \in T$ be such that F is T plus a $w - u$ path. There is (at least one) i such that w is in a $u_i -$

u_j and in a $u_i - u_k$ path (again, $\{i, j, k\} = \{0, 1, 2\}$). Again, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity, and at least one of these is what we search for. See the second case in Figure **??**. $\square$

Shortest cycle cover problem

We briefly remark a related problem: the *shortest cycle cover problem*. Given a bridgeless graph G we care about a collection of cycles that covers every edge of G *at least once*. We denote by $scc(G)$ the minimal total length of such collection. Jaeger's 8-flow gives easily a 4-cover by 7 cycles; it follows that $scc(G) \leq 4m$. This can be certainly improved; the best known general result is $scc(G) \leq \frac{5}{3}m$ (Jamshy and Tarsi). (Better results are known for some classes of graphs, in particular for cubic graphs.) It is conjectured that $scc(G) \leq \frac{7}{5}m$ and this would, if true, imply the CDC conjecture.