## An equivalent formulation of NZ flows

**Theorem 22** (Hoffman's Circulation Theorem)**.** *Let $G$ be a digraph, let $0 < a \le b$ be integers. Then the following are equivalent.*

1. *There is a $\mathbb{Z}$-flow $f$ on $G$ such that $a \le f(e) \le b$ for each edge $e$ of $G$.*
2. *There is a $\mathbb{R}$-flow $f$ on $G$ such that $a \le f(e) \le b$ for each edge $e$ of $G$.*
3. *For each $U \subset V(G)$ we have $\frac{a}{b} \le \frac{|\delta^+(U)|}{|\delta^-(U)|} \le \frac{b}{a}$.*

DUES

*Proof.* (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): take any set $U$. As "the net flow over each cut is zero", we have

$$\sum_{e\in\delta^+(U)} f(e) = \sum_{e\in\delta^-(U)} f(e)\,.$$

...

(3) ⇒ (1): We call a $\mathbb{Z}$-flow reasonable if $0 \le f(e) \le b$ for each edge $e$. Find reasonable flow that is optimal in the following sense:

- $m := \min\{f(e) : e \in E(G)\}$ is as large as possible;
- among flows with the same $m$ we choose the one with as few edges attaining $f(e) = m$ as possible.

We claim that the optimal reasonable flow does in fact satisfy $f(e) \ge a$ for every edge, which would prove (1). For contradiction, suppose there is an edge $e_0 = u_0v_0$ for which $f(e_0) = m < a$.

*Good edges* $e$:

- $f(e) < b$ and we use $e$ forward,
- $f(e) > m + 1$ and we use $e$ backward.

Either a $v_0u_0$-path of good edges OR a cut certifying it. … □

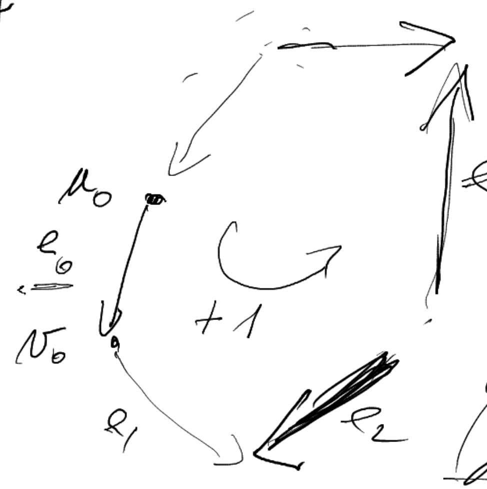
$v_0 \to u_0$

1) f je $\mathbb{Z}$-tok
2) $\forall e\ 0 \le f(e) \le b$
~~3) $\forall e\ f(e) \ge a$~~

$\exists f$, která splňuje 1) & 2): $f \equiv 0$

Pro spor

$f(e_0)$: $m \to m+1$




$f(e_0) = m < a$

$\boxed{f(e_3) < b}$

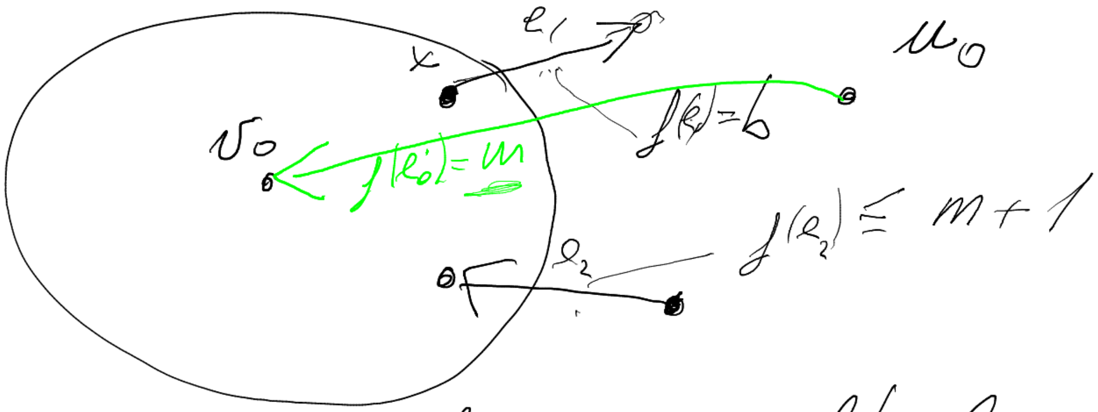
$\boxed{f(e_2) > m+1}$

~~$\ge m+1 \to u_1$~~

plán: zvýšit $f(e_0)$ o 1, tím spor s optimalitou

Pokud exist. ścieżka $v_0 \to u_0$ z dobrych krawędzi,

zwiększamy na $P + e_0$ wartość $f$ o $1$, ↯




$$\sum_{e \in \delta^+(X)} f(e) = \sum_{e \in \delta^-(X)} f(e)$$

$$|\delta^+(X)| \cdot b < |\delta^-(X)| \cdot \underbrace{(m+1)}_{\le a}$$

$X = \{ x : \exists$ ścieżka $v_0 \to x$ z dobrych krawędzi $\}$

$$\frac{a}{b} \le \frac{|\delta^+(X)|}{|\delta^-(X)|} < \frac{a}{b}$$

↯

# Cirkulární barevnost

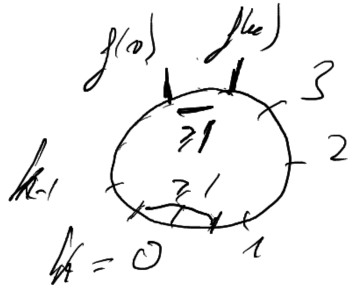
barvení grafu G: $f: V(G) \to \{0, 1, \dots, k-1\}$ : $f(u) \neq f(v)$ $\forall uv \in E$

$|f(v) - f(u)| \geq 1$

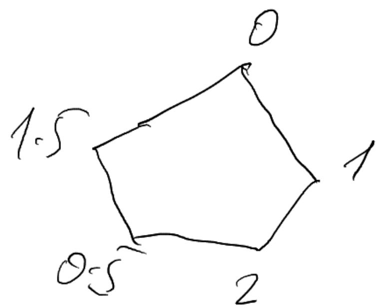
cirkulární barvení G: $f: V(G) \to [0, k) \subseteq \mathbb{R}$

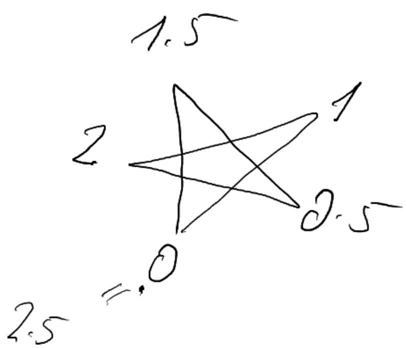
$\forall uv \in E: |f(u) - f(v) \bmod k| \geq 1$

$1 \leq |f(u) - f(v)| \leq k-1$



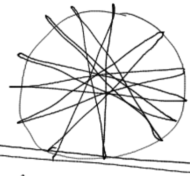

$G = C_5$




$k = 2.5$

$\chi_c(C_5) \leq 2.5$



$$\boxed{\chi_c(C_{2k+1}) = 2 + \frac{1}{k}}$$

$\chi_c(G) = \min\{k : \text{ex. cirk. } k\text{-barvení}\}$

$\chi(G) = \lceil \chi_c(G) \rceil$

## Circular flows

**Definition 23.** *Let $G$ be a digraph, $f$ a $\mathbb{R}$-flow, $r \in \mathbb{Q}$. We say that $f$ is* nowhere-zero circular $r$-flow, *if*

$$f(e) \in [1, r-1]$$

*for all edges $e \in E(G)$.*

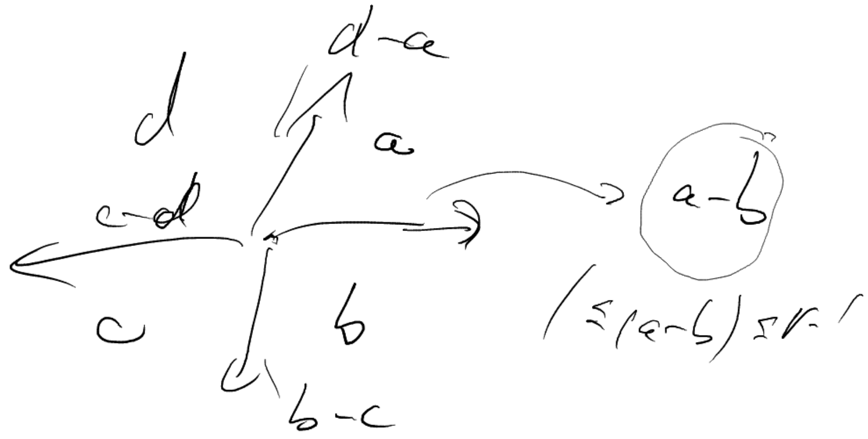
**Definition 24.** *Let $G$ be a digraph, $f$ a $\mathbb{Z}_q$-flow, $p, q \in \mathbb{N}$. We say that $f$ is* nowhere-zero circular $p/q$-flow, *if*

$$f(e) \in \{q, q+1, \ldots, p-q\}$$

*for all edges $e \in E(G)$.*

- Definition 23 and 24 are equivalent (for $r = p/q$).
- A variant of the circulation lemma for real $a$, $b$ also true (use just (2) and (3)).
- It follows that $k$-flow implies existence of $k'$-flow for all $k' > k$.






$$q \cdot \{1, 1+\tfrac{1}{q}, \ldots, \tfrac{p}{q}-1\}$$

$$\lceil \phi_c(G) \rceil = \phi(G)$$

$$\phi_c(G) = \chi_c(G^*)$$ pro $G$ rovinné nakreslené

## Snarks

A graph is called a *snark*, if it is

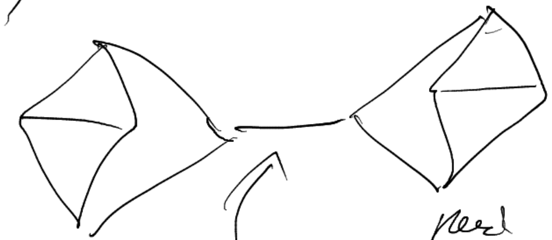
$3 \le \chi' \le 4$ (Brooks)

- cubic,
- bridgeless and
- not 3-edge-colorable.

Equivalently, it has no 4-NZF.

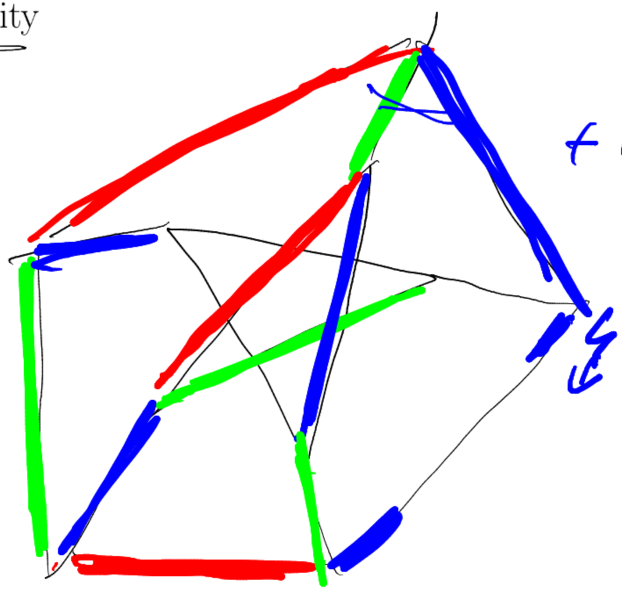
Some authors require a higher edge-connectivity (we may insist on the graph to be cyclically 6-edge-connected), but we won't do it here.

Snarks are canonical counterexamples.



není 3-ob.

není snark

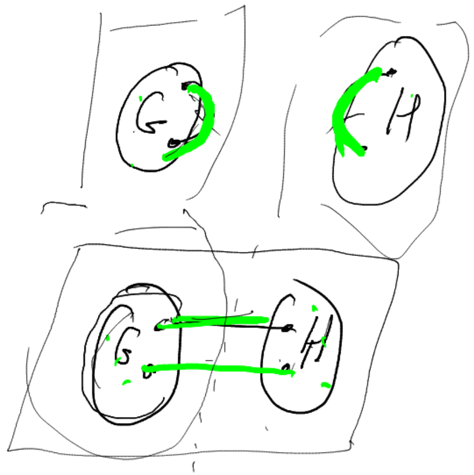


+ důkaz Petersenův graf

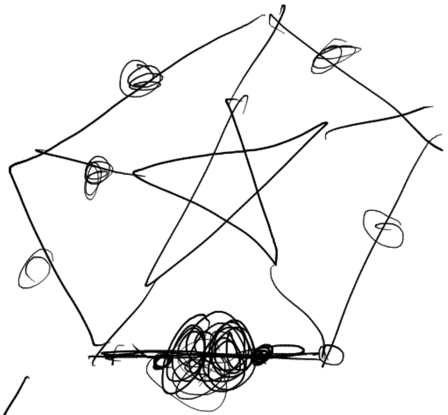
[illegible]

nejmenší snark

## Snarks with a 2-cut



Start with graphs $G$ and $H$ each with a specified edge. To form the graph $G{=}H$ we cut the specified edges in $G$ and $H$ and glue the "half-edges" to connect $G$ and $H$.



$$\boxed{G{=}H \text{ is a snark} \iff G \text{ or } H \text{ is a snark.}}$$

$\Rightarrow$

$\neg \Leftarrow \neg$ G i H mají 3-ob.

Equivalently: when we "add anything to an edge of a snark", we get again a snark and all snarks with a 2-cut are obtained this way.

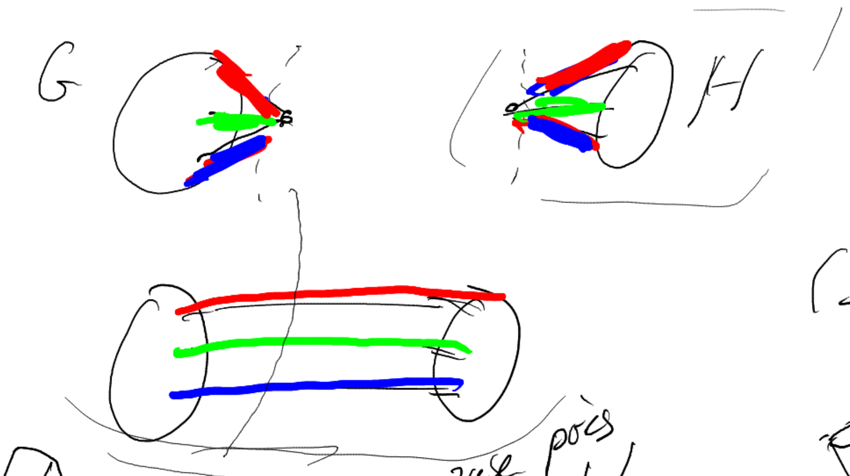
$\neg \Rightarrow \neg$ G=H má 3-ob. $\Downarrow$ obě hrany v řezu mají stejnou barvu

$\mathbb{Z}_2^2$ - NZF (0,1) (1,0) (1,1)

Any edge 3-coloring of $G{=}H$ gives the same color to the two edges of the 2-cut. Consequently, we may use the coloring of $G{=}H$ to get colorings of $G$ and of $H$. OTOH …

## Snarks with a 3-cut

Now we start with cubic graphs $G$, $H$ each with one specified vertex. We split these specified vertices in three vertices of degree 1, and identify the three pendant of $G$ with those of $H$. (There are 3! ways to do so.) We use $G \equiv H$ to denote the resulting graph.




(3! možností)

| |
|---|
| $G \equiv H$ is a snark $\iff$ $G$ or $H$ is a snark. |

Dk $\neg \Rightarrow \neg$

zde počs tady

$G \equiv H$ má 3-ob. $\Rightarrow$ hrany v 3-řezu mají různé 3 barvy

$\Rightarrow$ 3-ob. v $G$ i v $H$

Equivalently: when we “add anything to a vertex of a snark”, we get again a snark, and all snarks with a 3-cut are obtained this way.

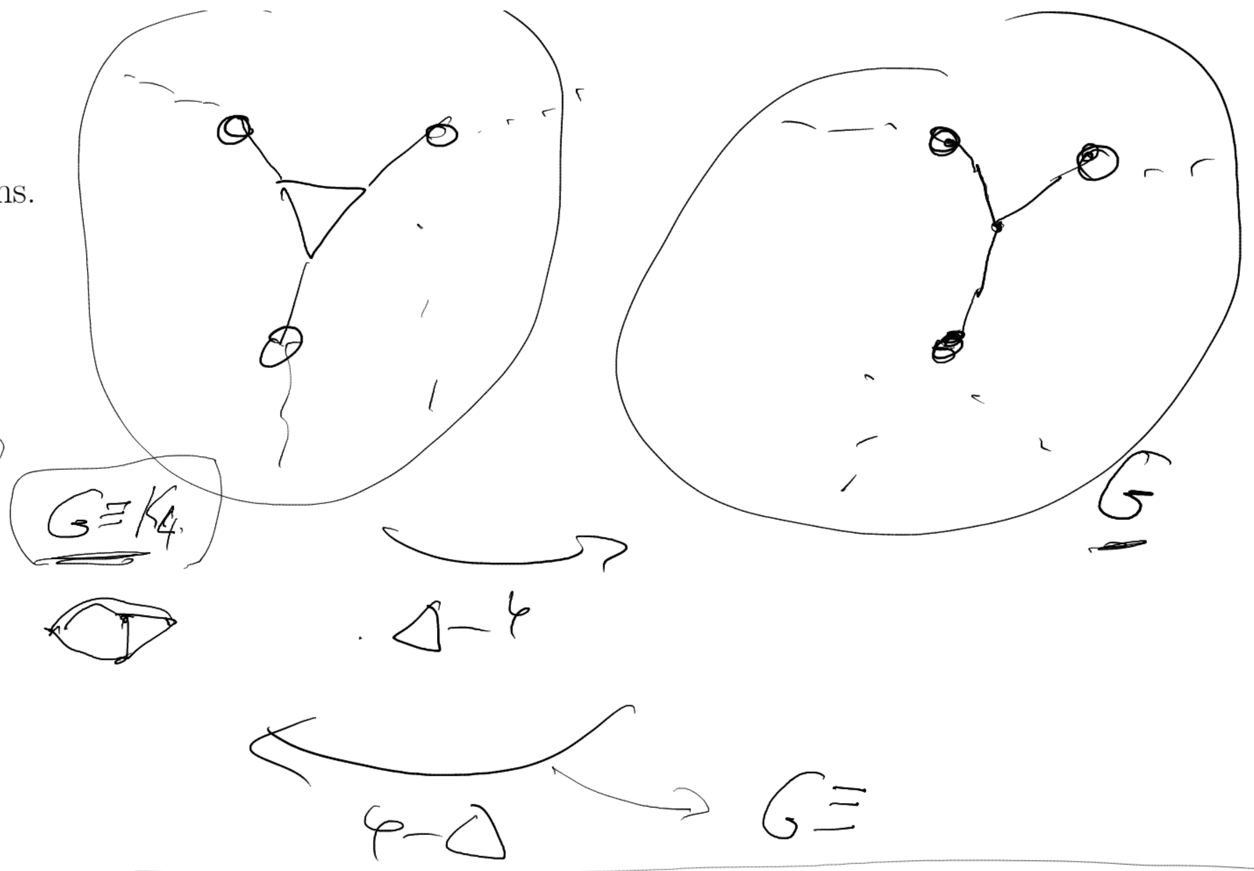
$\neg \Leftarrow \neg$ $G$ i $H$ mají 3-ob. ze sym. barev

3 symetrie barev . . . jde obarvení slepit

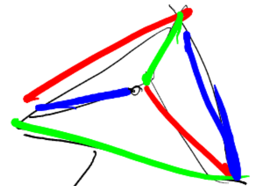
## Exercise

We define two useful operations on cubic graphs. A $\Delta$-Y transformation is a contraction of a triangle to a single vertex, a Y-$\Delta$ transformation is the inverse operation. (Observe that these operation preserve the 3-regularity.) For a cubic graph $G$, prove that $G$ is a snark iff $G'$ obtained by a series of Y-$\Delta$ and $\Delta$-Y transformations from $G$ is a snark.

Note: The simplicity of the above two constructions, in particular the fact that only one of the smaller graphs needs to be a snark, together with possibility to reduce the "big conjectures" to cyclically 4-edge-connected graphs, explain why some authors choose to demand that the snarks are free of 2-cuts and nontrivial 3-cuts.




$K_5$ není snark



$G$ snark $\iff$ $G \cong K_5$ je snark

## Snarks with a 4-cut – Isaacs' dot product

Let $G$, $H$ be graphs, $ab$, $cd$ edges of $G$, $e$ an edge of $H$, let $x, y$ be the other two neighbours of one end of $e$, $u, v$ the other two neighbours of the other end. To form the *Isaacs' dot product* $G \cdot H$ of $G$ and $H$ we delete edges $ab$ and $cd$ from $G$, $e$ with its end-vertices from $H$, and add edges $ax$, $by$, $cu$, $dv$.

**Theorem 25** (Isaacs, 1975)**.** *If G and H are snarks then so is $G \cdot H$. If both G and H are cyclically 4-edge-connected and if the vertices $a, b, c, d$ are all different, then $G \cdot H$ is also cyclically 4-edge-connected.*

*Proof.* Suppose we have an edge 3-coloring $f$ of $G \cdot H$. We distinguish two cases.

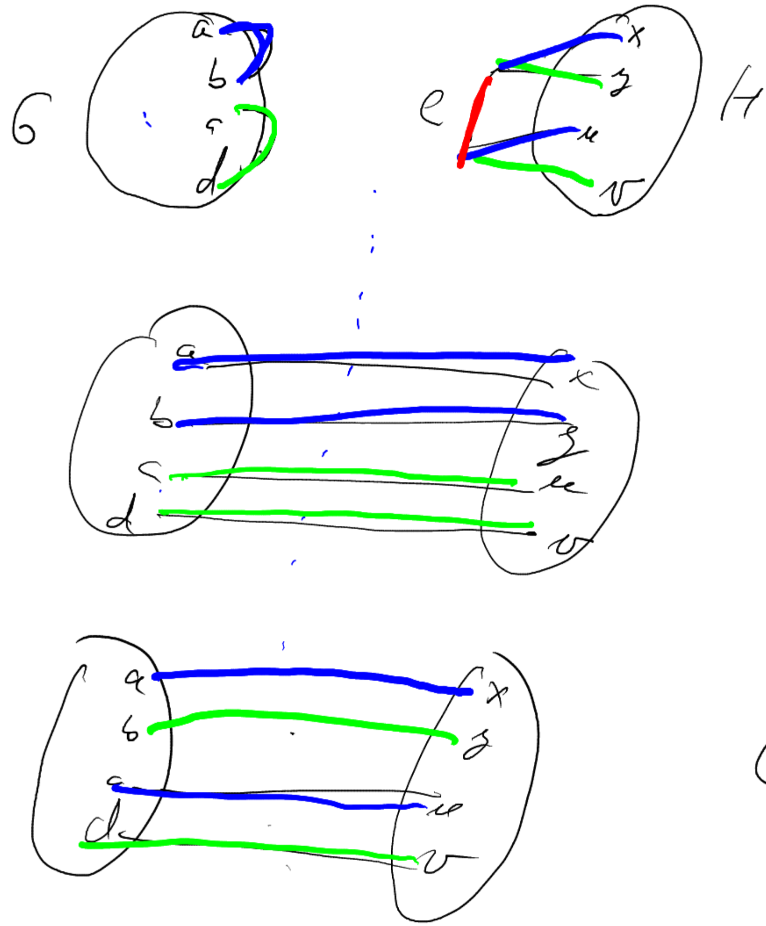
(1) $f(ax) = f(by)$ (2) $f(ax) \neq f(by)$

...

G není snark

H není snark

□

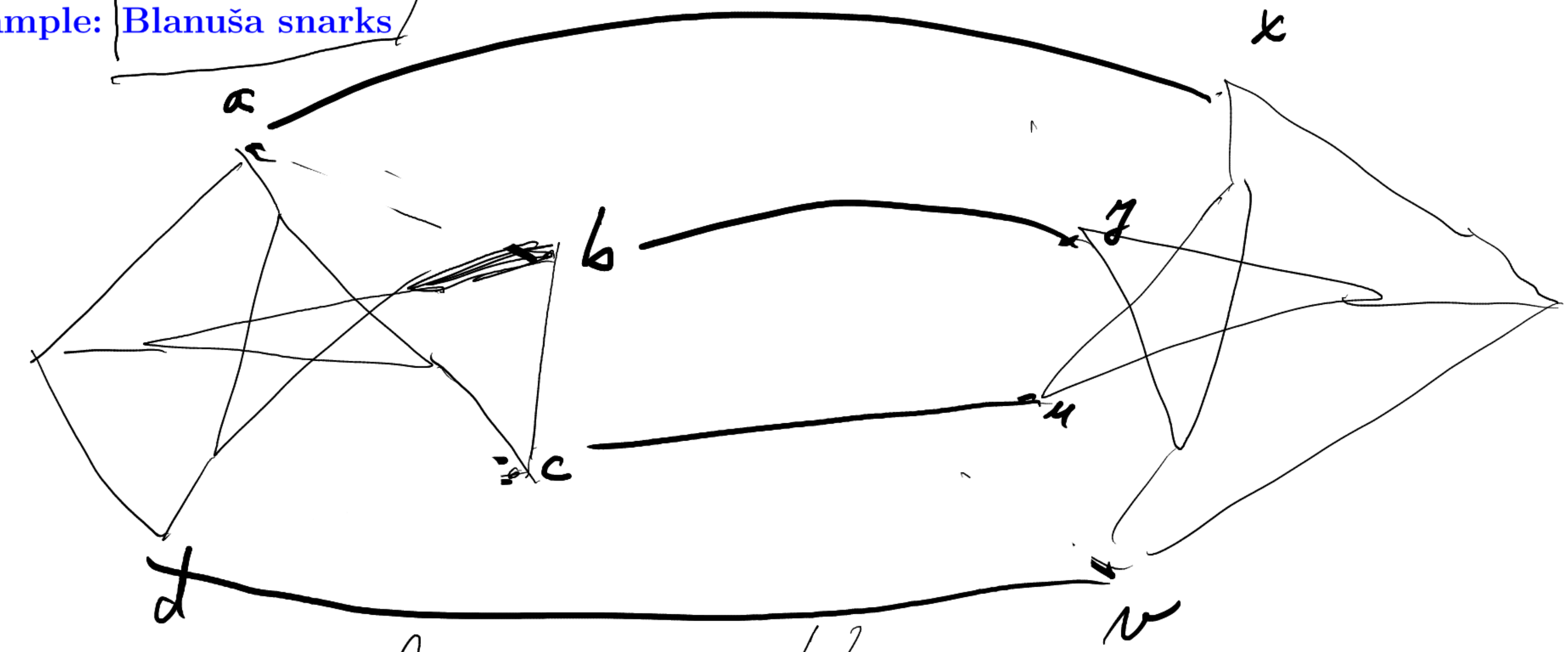



G·H snark

⇑

G,H snarky

## Example: Blanuša snarks

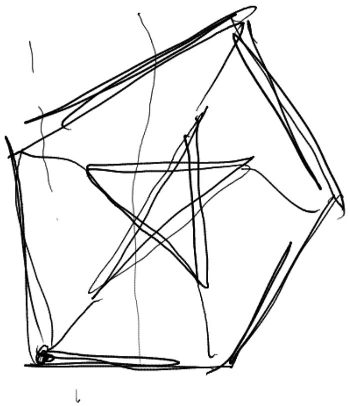



3-reg. graf s 18 vrcholy

G je $k$-souvislý $\iff \forall U \subsetneq V(G),\ U \neq \emptyset$ $\quad |\delta(U)| \geq k$

G je interně $k$-souvislý $\iff \forall U$ $\quad |\delta(U)| \geq k$ nebo $|U| = 1$ nebo $|\overline{U}| = 1$

G je cyklicky $k$-souvislý $\iff \forall U$ $\quad |\delta(U)| \geq k$ nebo $G[U]$ je les nebo $G[\overline{U}]$ je les



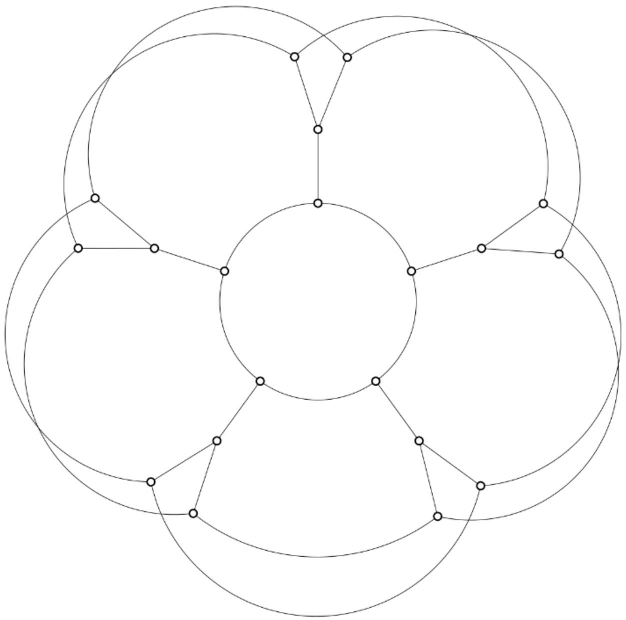
2-souvislý

3-souvislý

není 4-souvislý $\quad$ je interně/cykl. 4-souvislý

cykl. 5-souvislý

## Flower snarks



PŘÍŠTĚ

Let $n$ be odd. To describe a graph $J_n$, we start with three copies of $C_n$, we denote its vertices by $i_1$, $i_2$, $i_3$ for $i = 1, \ldots, n$. Replace edges $n_2 1_2$ and $n_3 1_3$ by $n_2 1_3$ and $n_3 1_2$. Finally, for each $i$ we add a new vertex $i$ and join it by an edge to $i_1$, $i_2$, $i_3$. On Figure **??** we can see $J_5$ (this particular graph is sometimes called the flower snark). and $J_3$ — is just a Y-$\Delta$ transformation of $Pt$ (equivalently, it is Pt$\equiv K_4$).

**Theorem 26** (Isaacs, 1975)**.** *If $n$ is odd then $J_n$ is a snark. If $n \geq 7$ then $J_n$ is cyclically 6-edge-connected.*

*Proof.* Suppose $J_n$ can be edge-colored using three colors. Let $B_i$ denote the subgraph induced by vertices $i, i_1, i_2, i_3$ and the incident edges (see Fig. **??**). We divide the edges of this subgraph into three triples, Left, Right, and Top. (Of course the Right edges of $B_i$ are the Left edges of $B_{i+1}$.) Clearly not all edges of of $L$ can be of the same color, as then it is not possible to color $T$. Thus there are two possibilities.

**(1) Edges of $L$ use one color twice.** Say, they use colors 1, 1, and 2 in some order. It is easy to check that then edges of $R$ use colors 2, 3, and 3, in some order. In the next block we will use 1, 1, 2 on the right, and so on. As $n$ is odd, we get a contradiction.

**(2) Edges of $L$ use all three colors.** Again, it is simple to explore the two possibilities how to extend the coloring on $R$: both are obtained from the coloring of $L$ by a cyclic shift (i.e., a permutation formed by one 3-cycle). In between the blocks $B_n$ and $B_1$ we introduced a transposition by the construction of the graph. Thus if there is an edge 3-coloring, then we can write an identity as a composition of 3-cycles and one transposition, which is a contradiction.

TODO: cyclic connectivity?

□