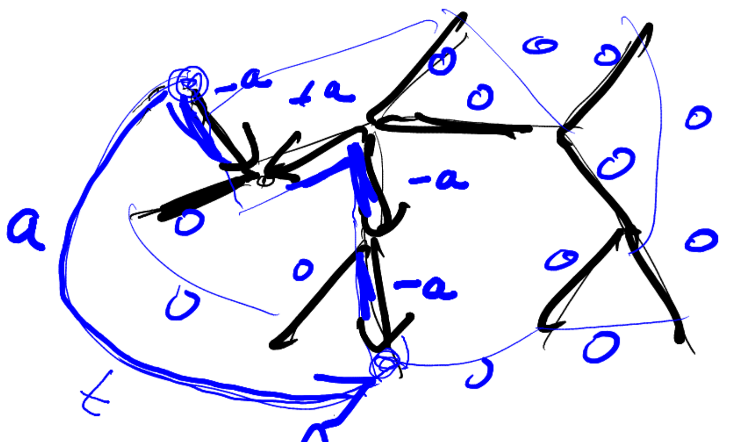
(Recall)

Flows and spanning trees – sum

Let T be a spanning tree of G. Now for every edge $t \in E(G) \setminus E(T)$ and every $a \in \Gamma$ we let $\varphi_{t,a}$ be the (unique) flow in G such that

- $\varphi_{t,a}(t) = a$
- $\varphi_{t,a}(e) = 0$ for $e \neq t$ and $e \in E(G) \setminus E(T)$

… *elementary flow with respect to T.*



elementární kružnice

- $\mathcal{F}_\Gamma(G)$:= the vector space of all flows $\subset \Gamma^{E(G)}$
- (we need Γ to be a field).
- For any fixed spanning tree T the elementary flows $\{\varphi_{t,1} : t \in E(G) \setminus E(T)\}$ form a basis of $\mathcal{F}_\Gamma(G)$.
- Any mapping $\varphi : E(G) \setminus E(T) \to \Gamma$ can be uniquely extended to a Γ-flow on G.
- No control over the edges of T, thus we can't use this easily to construct a NZ flow.

v $\Gamma^{E(G)}$ je $\mathcal{F}_\Gamma(G)$ podprostor určený $|V(G)|$ rovnostmi (Kirch. zák.)

$$\sum_{t \in E(G)\setminus E(T)} \underbrace{\varphi(t)}_{\in \Gamma} \cdot \underbrace{\varphi_{t,1}}_{\text{tok}} \text{ — tok}$$

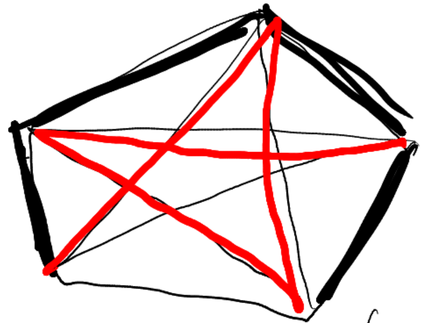
skládání koster

$S\varphi$ nic o kostře → všude

(Recall)

Flows and spanning trees – product

Theorem 17. *Any 4-edge connected graph admits a $\mathbb{Z}_2^2$-NZF.*



Proof. If G is 4-edge connected, then there are two disjoint spanning trees, T_1 and T_2 (proof later).

Let f_i be the $\mathbb{Z}_2$-flow on G that equals 1 on all edges not in T_i. (Such flow exists — see above.)

Now put $f = (f_1, f_2)$. This is indeed a $\mathbb{Z}_2^2$-flow, and if $f(e) = 0 = (0,0)$ for some edge e then e lies in both T_1 and T_2, a contradiction. □

f_i: ... $\mathbb{Z}_2$-tok równy 1 poza $E(T_i)$

	$E(T_1)$: $e_1\ e_2 \dots$	$E(T_2)$: $e_k \dots$	zbytek
f_1	? ? 0/1 ... ?	1 1 ... 1	1 ... 1
f_2	1 1 1 1	? ... ?	1 ... 1

$f = (f_1, f_2) : E(G) \to \mathbb{Z}_2^2 \setminus \{(0,0)\}$

tok

□

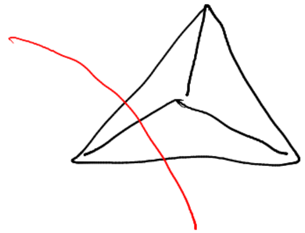
Theorem 18 (Jaeger)**.** *Any bridgeless graph admits a* $\mathbb{Z}_2^3$*-NZF.*

Proof. $\boxed{\text{Suppose first that } G \text{ is 3-edge connected,}}$ we will use spanning trees similarly as in the construction of a NZ 4-flow.

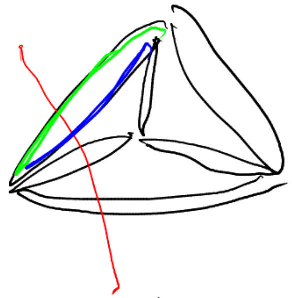
We let G' be the (multi)graph obtained from G by adding to each edge a new one, parallel to it.

G' is 6-edge connected ...

$\boxed{\text{START}}$



G



G'

$\exists\, T_1, T_2, T_3$ kostry v G', tzv. disj.

~~$\exists\, T_1, T_2, T_3$ — " — v G~~

$E(T_1) \cap E(T_2) \cap E(T_3) = \emptyset$

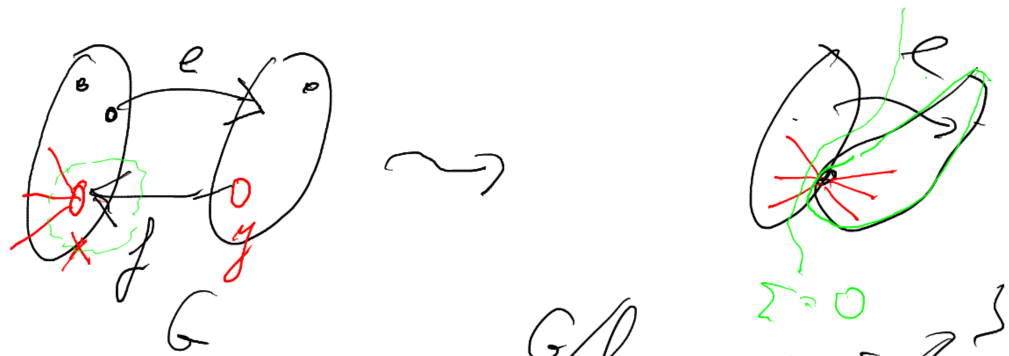
T_i na $\mathbb{Z}_2$-tok φ_i v G : $\underline{\varphi_i(e) = 1}$ $\forall e \notin E(T_i)$

$\varphi := (\varphi_1, \varphi_2, \varphi_3)$ ... $\mathbb{Z}_2^3$-tok v G

$\varphi(e) = (0,0,0) \Rightarrow e \in E(T_i)\ \forall i \Rightarrow$ ↯

So the theorem holds for all 3-edge-connected graphs. To prove it for all bridgeless graphs, suppose there is a counterexample and choose one with minimal number of edges, let it be denoted G. . . .

$\square$



G/f — má $\mathbb{Z}_2^3$-NZF

φ

$\varphi(f) = \varphi(e)$

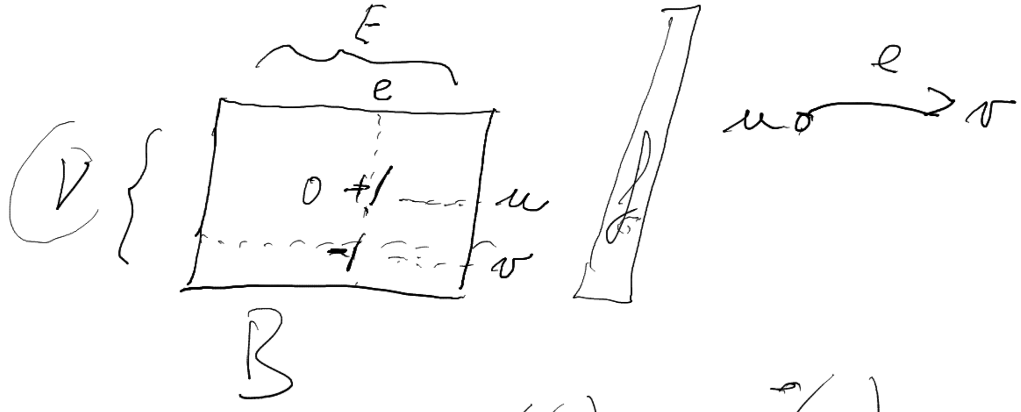
je $\mathbb{Z}_2^3$-NZF

v G

Flows and tensions:
a linear-algebra point of view

Let $G = (V, E)$ be a digraph. The *incidence matrix* of G is $B = (B_{v,e})_{v \in V, e \in E}$ defined by $B_{v,e} = +1$ if e starts at v, $B_{v,e} = -1$ if e ends at v, and $B_{v,e} = 0$ otherwise.

- Γ^E – set of mappings / vector space
- for every $f \in \Gamma^E$ the product Bf has the v-coordinate equal to $f^+(v) - f^-(v)$.
- flows = ker B



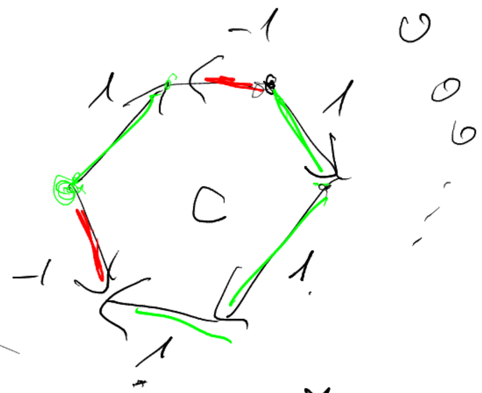
$(Bf)_v = f^+(v) - f^-(v)$

Tensions

Let $t : E \to \Gamma$ be a mapping. We say that t is a *tension* whenever

$$\sum_{e \in C^+} t(e) = \sum_{e \in C^-} t(e) \qquad (1)$$

holds for every circuit C with C^+ and C^- being the edges oriented forward and backward, respectively, along C.



$(t, c) = 0$

Motiv.: elektr.

$t(u,v)$ … o kolik stoupne napětí mezi u a v

- "does not depend on orientation"
- For any $F \subseteq E$, t is a tension in G whenever t_F is a tension in G_F.
- an easy way to get tensions: *potential difference*.
- For $p : V \to \Gamma$ we define its difference $\delta p : E \to \Gamma$ by letting $(\delta p)(u, v) = p(v) - p(u)$.
- δp is a tension for every p.

- $\langle f, g\rangle := \sum_{e\in E} f(e)g(e)$.
- not in general an inner product! (Why?)
- Still, many notions from linear algebra generalize easily. In particular, t is orthogonal to all elements of a vector space, whenever it is orthogonal to all elements of a generating set.
- Equation (1) $\iff$ $\langle t, c\rangle = 0$ for a particularly simple flow c: one that is zero outside of a circuit C and has values ± 1 on C.
- We already know that circuits generate $\mathcal{F}$ the space of all Γ-flows on G. (Why?)
- Thus, t is an element of $\mathcal{F}^\perp$.

- není def. v tgrupě
- sk. součin $\langle f,g\rangle = \sum f(e)\cdot\overline{g(e)}$
- $\langle f,f\rangle \geq 0 \quad \forall f$

  $\langle f,f\rangle = 0$ jen pro $f=0$

TADY NEPLATÍ!

$X \subseteq \Gamma^E$

$X^\perp = \{ y \in \Gamma^E : \langle x,y\rangle = 0 \}$

$\Gamma = \mathbb{Z}_k / \mathbb{Z}$

$T \subseteq \mathcal{F}^\perp$

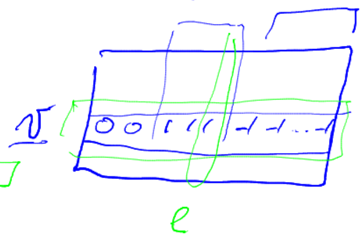
Theorem 19. *Let $\mathcal{F}$, $\mathcal{T}$ be the vector spaces (or modules) of all flows and all tensions, respectively, defined on a digraph G. Then*

$$\mathcal{F}^{\perp} = \mathcal{T} \qquad \text{and} \qquad \mathcal{T}^{\perp} = \mathcal{F}\,.$$

Moreover, $\mathcal{F} = \ker B$ and $\mathcal{T}$ is the row space of B, the icidence matrix of G.

$\mathcal{F} = \ker(B)$

$\mathcal{T} = R(B)$



Consequently

$$\boxed{\text{for every tension } t \text{ there is a potential } p \text{ such that } t = \delta p.}$$

δp

$p: V \to \mathbb{R}$

Indeed, δp can be expressed as $B^T p$. And, obviously, $B^T p$ is a general form of a linear combination of rows of B.

$t \in \mathcal{T}$

$t = \delta p$

$t = p^T \cdot B$

$(p^T B)_e = p(v) - p(u)$

$e = uv$

$\Rightarrow$ LA:

$$R(B)^\perp = \ker B = \mathcal{F}$$

$$\mathcal{J} \supseteq R(B) = (\ker B)^\perp$$

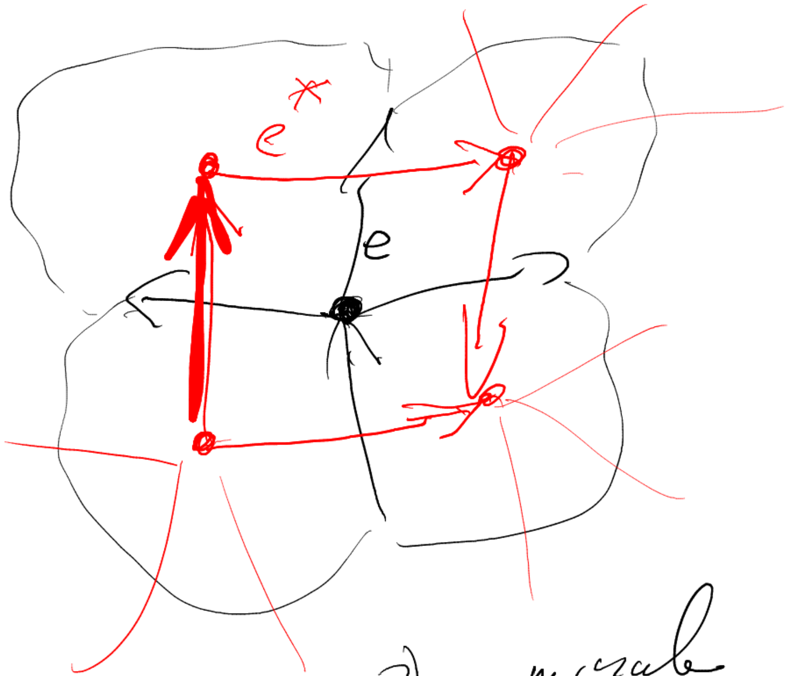
$$\mathcal{F}^\perp \subseteq \mathcal{J} \subseteq \mathcal{F}^\perp$$

rovnost

$$\Rightarrow \mathcal{J} = R(B)$$

Flows and tensions for a planar graph

- G: a plane digraph (planar digraph with a fixed drawing in the plane).
- *dual graph* G^* is a digraph with vertices being the faces of G and with edges corresponding to edges of G: the edge e^* connects the face on the left of e to the face on the right of e.
- $(G^*)^* = ?$ G s obrác. šipkami



Theorem 20. *The following is equivalent for a plane graph G:*

1. G has a NZ k-flow

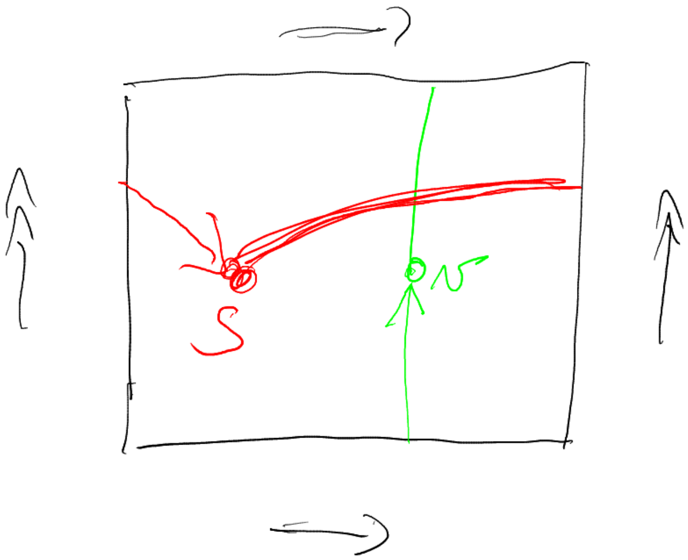
2. G^* has a NZ k-tension

3. G has a proper face-coloring by k colors.

1) $\Leftarrow$? 3) mícrale $p: V(G^*) \to \Gamma$ štěn. obarvení

2) $\exists\, t = \delta p$ .... $t(e) \neq 0$ $\forall e$

($\forall t$ $\exists p$)



$S\Omega$ v G^* ..... nemá dobré ob.

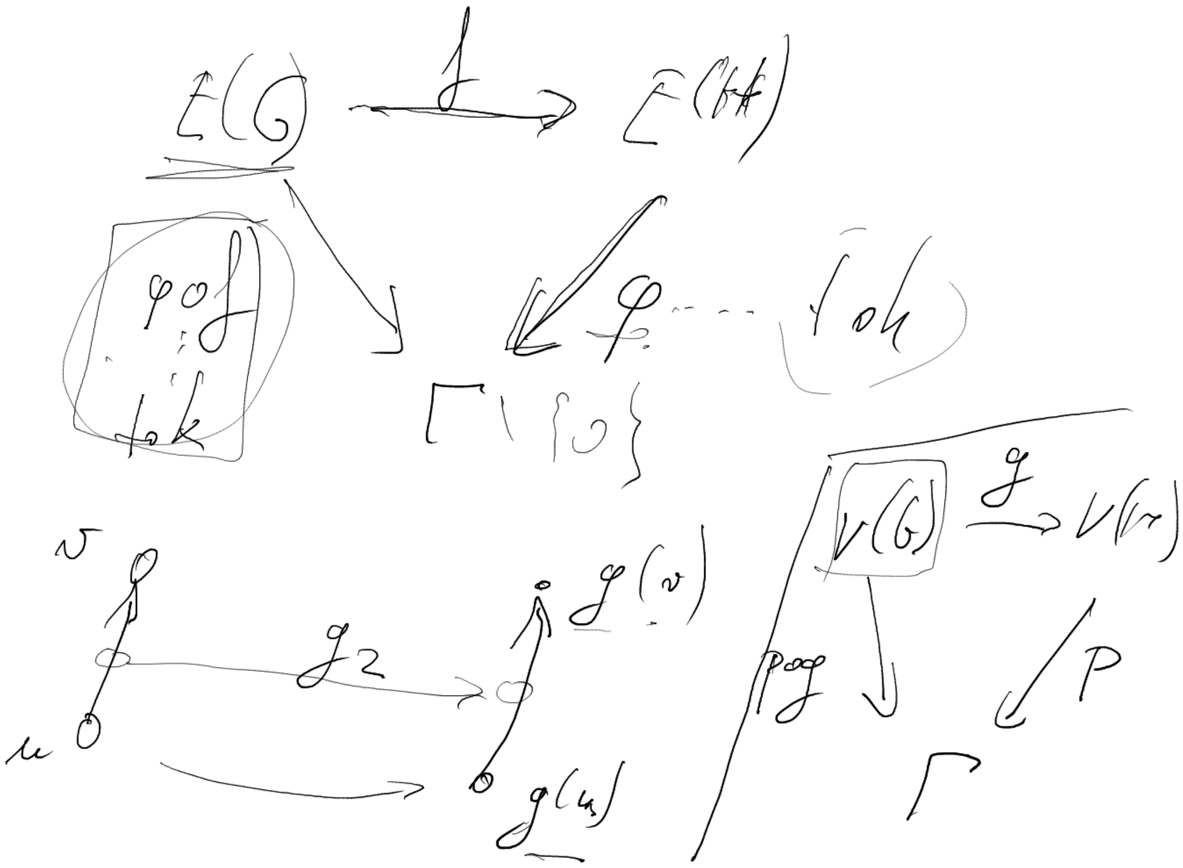
nemá nenul. tenzí

G ..... má rt hol v lib. Γ

Further abstraction possible

- A mapping $f : E(G) \to E(H)$ is Γ-*flow continuous* iff for every Γ-flow on H, $\varphi \circ f$ is a Γ-flow on G.
- A mapping $f : E(G) \to E(H)$ is Γ-*tension continuous* iff for every Γ-tension on H, $\varphi \circ f$ is a Γ-tension on G.
- Motivation: can help to solve flow-related problems:

Theorem 21. *If $g : V(G) \to V(H)$ is a graph homomorphism that the induced mapping on edges $((u, v) \mapsto (g(u), g(v)))$ is Γ-tension continuous for every Γ*



g_2

Dk: $\varphi \ldots$ Γ-tenze na H

$\varphi = \delta p \qquad p : V(H) \to \Gamma$

stačí ověřit: $g_2(\psi) = \varphi$

$p \circ g : V(G) \to \Gamma$

$\psi = \delta(p \circ g)$ je Γ-tenze slovníček pro přechod do Hoceologie

tok ... proto prostoru cyklů

fáze ... proto prostoru kohranic

An equivalent formulation of NZ flows

Theorem 22 (Hoffman's Circulation Theorem)**.** *Let G be a digraph, let $0 < a \le b$ be integers. Then the following are equivalent.*

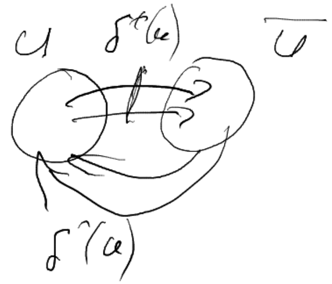
1. *There is a $\mathbb{Z}$-flow f on G such that $a \le f(e) \le b$ for each edge e of G.*
2. *There is a $\mathbb{R}$-flow f on G such that $a \le f(e) \le b$ for each edge e of G.*
3. *For each $U \subset V(G)$ we have $\frac{a}{b} \le \frac{|\delta^+(U)|}{|\delta^-(U)|} \le \frac{b}{a}$.*

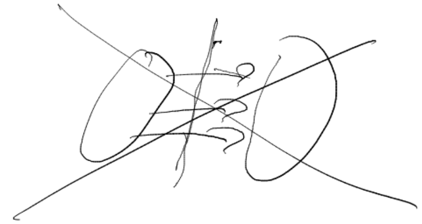
Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): take any set U. As "the net flow over each cut is zero", we have

$$a|\delta^+(U)| \le \sum_{e\in\delta^+(U)} f(e) = \sum_{e\in\delta^-(U)} f(e) \le \sum_{e\in\delta^-(U)} b = b\cdot|\delta^-(U)|$$

$$b\cdot|\delta^+(U)| \ge \sum_{e\in\delta^+(U)} f(e), \qquad \sum_{e\in\delta^-(U)} f(e) \ge a\cdot|\delta^-(U)|$$

$$f^+(U) = f^-(U)$$





$$1 \Rightarrow 2 \Rightarrow 3$$

$$\frac{a}{b} \le \frac{|\delta^+(U)|}{|\delta^-(U)|} \le \frac{b}{a}$$

(3) $\Rightarrow$ (1): We call a $\mathbb{Z}$-flow reasonable if $0 \leq f(e) \leq b$ for each edge e. Find reasonable flow that is optimal in the following sense:

- $m := \min\{f(e) : e \in E(G)\}$ is as large as possible;
- among flows with the same m we choose the one with as few edges attaining $f(e) = m$ as possible.

We claim that the optimal reasonable flow does in fact satisfy $f(e) \geq a$ for every edge, which would prove (1). For contradiction, suppose there is an edge $e_0 = u_0v_0$ for which $f(e_0) = m < a$.

Good edges e:

- $f(e) < b$ and we use e forward,
- $f(e) > m + 1$ and we use e backward.

Either a u_0v_0-path of good edges OR a cut certifying it. ... □

NEXT TIME

Fractional flows

Definition 23. *Let G be a digraph, f a $\mathbb{Z}_q$-flow, $p, q \in \mathbb{N}$. We say that f is* nowhere-zero fractional p/q-flow, *if*

$$f(e) \in \{p, p+1, \ldots, q-p\}$$

for all edges $e \in E(G)$.

- A variant of the circulation lemma for real a, b also true (use just (2) and (3)).
- It follows that k-flow implies existence of k'-flow for all $k' > k$.

NEXT TIME