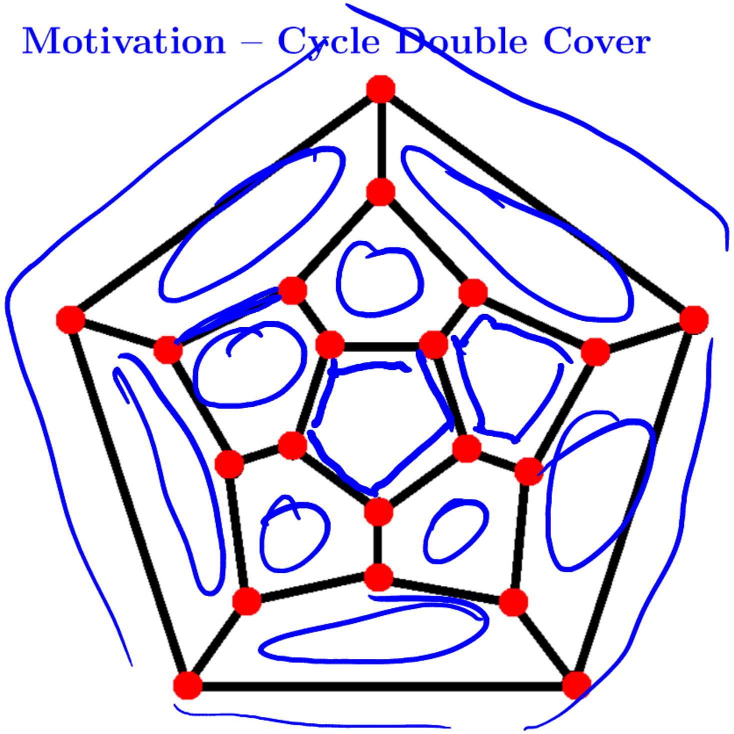
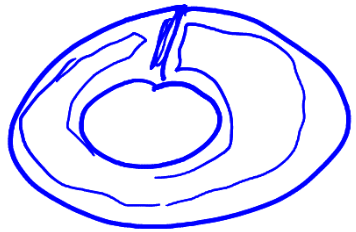
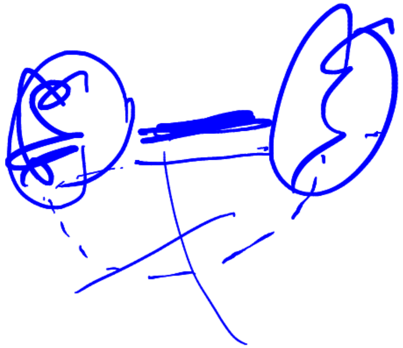
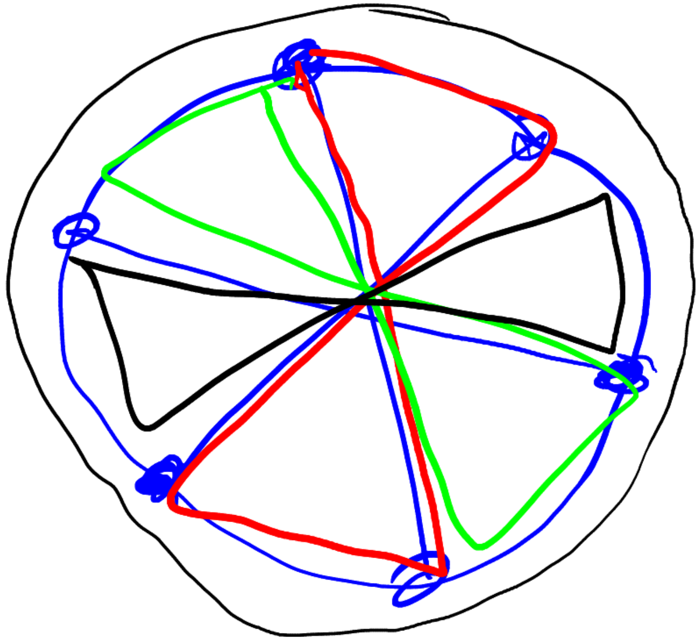
## Motivation – Cycle Double Cover







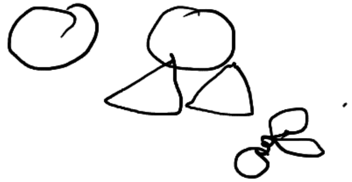


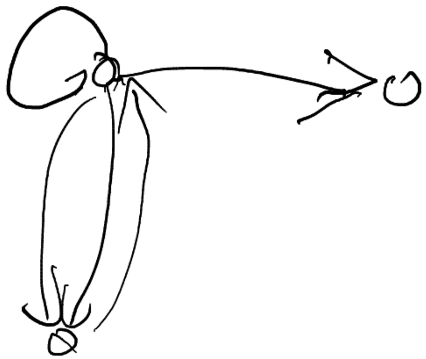
for planar bridgeless graphs the face-boundaries are a collection of circuits that cover every edge exactly twice. What about nonplanar?

CDC

## Definitions

- circuit (kružnice) := 2-regular connected graph (subgraph of another graph)
- cycle (cyklus) = even graph = eulerian graph := edge disjoint union of circuits
- digraph := directed multigraph, loops allowed
- group := abelian group





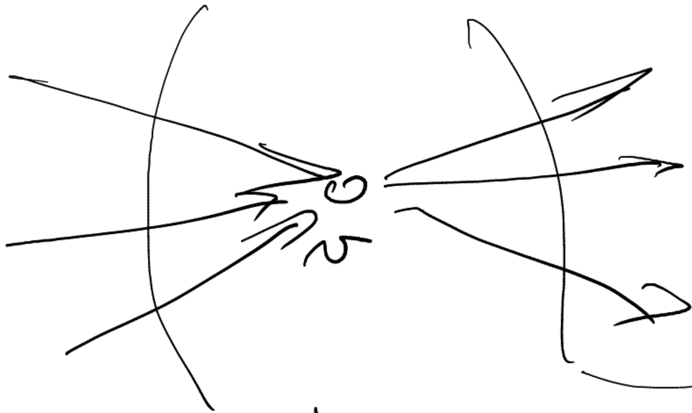
arc / edge

**Definition 1.** *Let $G$ be a digraph, $\Gamma$ a group. A mapping $f : E(G) \to \Gamma$ is called a* flow *(or, more explicitly, a $\Gamma$-flow), if for every vertex $v \in V(G)$ the Kirchhoff law is valid:*

$$\sum_{e=(v,u)} f(e) = \sum_{e=(u,v)} f(e)\,.$$

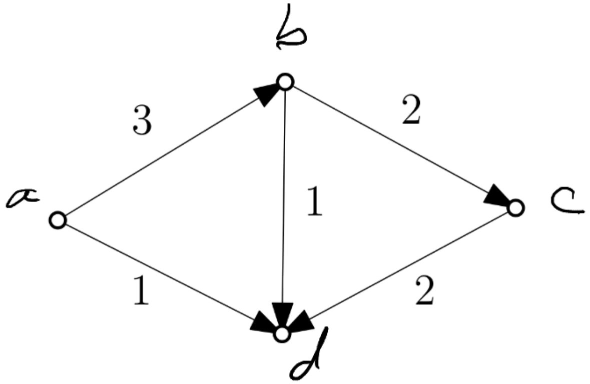
$f^+(v)$ = *the left-hand side of the above equation, the amount of flow that leaves $v$,*

$f^-(v)$ = *the right-hand side of the above equation, the amount of flow that enters $v$.*




$f^+(v)$

$$\sum_{e=(u,v)} f(e) = f^-(v)$$




$f^+(b) = 1+2$

$f^-(b) = 3$

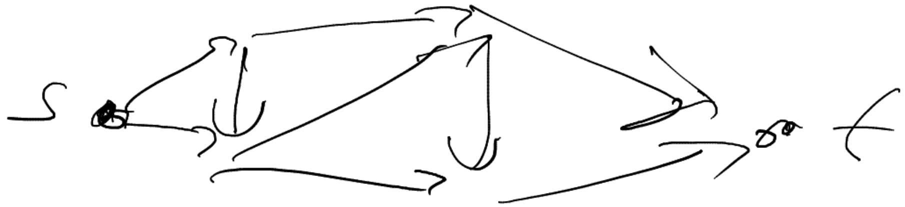
$f^+(a) = 1+3$

$f^-(a) = 0$

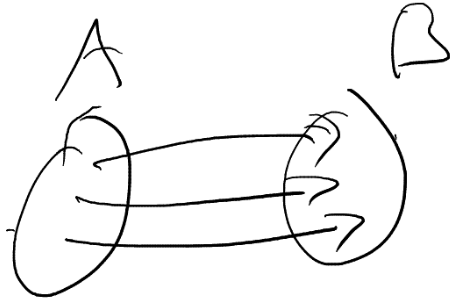
$\Gamma = \mathbb{Z}_4$

- $f \equiv 0$ is a flow.
- if $f$, $g$ are flows, then $f \pm g$ are also flows
- the set of all $\Gamma$-flows on a given digraph is again an (abelian) group.
- If $\Gamma$ is a field, than the set of all $\Gamma$-flows is a vector space.
- related notion – flows in networks.
- $\mathbb{R}^d$-flow. The same definition. Esp. for $d = 3$ has a meaning in physics: momentum-preservation, Feynmann diagrams.

podgrupa $\Gamma^{E(G)}$




$\Gamma = \mathbb{R} \,/\, \mathbb{Q}$




**Notation** $A$, $B$ are sets of vertices

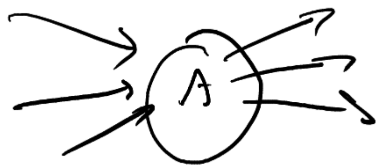
$f(A, B) = \sum f(e) : e$ starts in $A$ and ends in $B$

$f^+(A) = f(A, \bar{A})$

$f^-(A) = f(\bar{A}, A)$

(where $\bar{A} = V(G) \setminus A$).

$f(A)$




$f^+(A)$

**Observation 2.** *Let $G$ be a digraph, $\Gamma$ a group, $f$ a $\Gamma$-flow. Then for $A \subseteq V(G)$*

$$f^+(A) = f^-(A)\,.$$

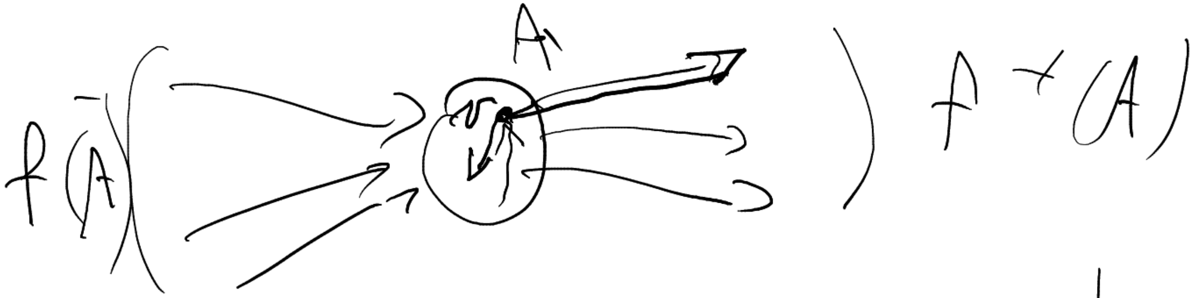
*Proof.* Let us sum the Kirchhoff law for all $v \in A$. □

**Corollary 3** (a flow and small cuts)**.** *Let $G$ be a digraph, $\Gamma$ a group, $f$ a $\Gamma$-flow.*

- *If $e$ is a bridge then $f(e) = 0$.*
- *If $e, e'$ form a 2-cut (and are oriented in the same direction) then $f(e) + f(e') = 0$.*

**Corollary 4** (a flow and a partition)**.** *Let $G$ be a digraph, $\Gamma$ a group, $f$ a $\Gamma$-flow. Consider any partition $\mathcal{P}$ of $V(G)$. Let $G_{\mathcal{P}}$ be the graph where each equivalence class is identified to a vertex and all edges are preserved and let $f_{\mathcal{P}}$ be the restriction of $f$ to edges of $G_{\mathcal{P}}$. Then $f_{\mathcal{P}}$ is a $\Gamma$-flow on $G_{\mathcal{P}}$.*

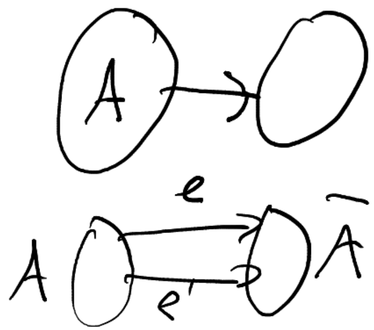





$$\sum_{v\in A} f^+(v) = f^+(A) + f(A,A)$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

$$\sum_{v\in A} f^-(v) = f^-(A) + f(A,A)$$

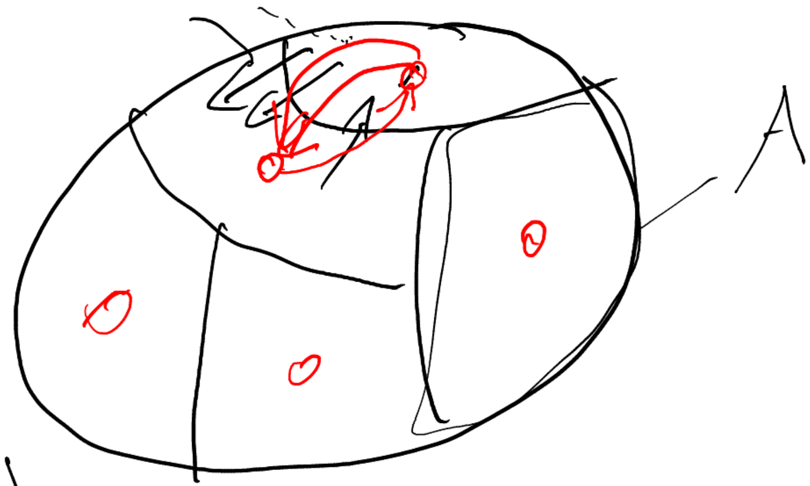



$$f^+(A) = f^-(A) = 0$$

$$f^-(A) = 0$$

$$\| \quad f^+(A) = f(e) + f(e')$$

$f^{f(A)} = f^{-}(A)$




- - - Kvd. z. pro $G_p$ a vrchol $A$

$V(G)$ $P$

$G/P = G_p$

## Nowhere-zero flows


**Definition 5.** *Let $G$ be a digraph, $\Gamma$ a group, $f$ a $\Gamma$-flow. We say that $f$ is a* nowhere-zero $\Gamma$-flow, *if $f(e) \neq 0$ for all edges $e \in E(G)$. Frequently we will shorten nowhere-zero to NZ.*

- bridge $\Rightarrow$ no NZ flow.
- the opposite is also true
- dependence on the group $\Gamma$.

na $\forall \Gamma$

#NZ $\mathbb{Z}_2$-toků = #NZ $\mathbb{Z}_k$-toků

tok : zobr. $E(G) \to \Gamma$

**Theorem 6** (flow polynomial,Tutte 1954)**.** *For every graph $G$ there is a polynomial $P_G(x)$ s.t. for every group $\Gamma$, the number of NZ $\Gamma$-flows on $G$ is $P_G(|\Gamma|)$.*

$\Gamma \neq \mathbb{Z}_1$

1) $E(G) = \emptyset$

ex. $\emptyset$ .... zobr. z $\emptyset \to \Gamma$

$|\Gamma|^0$ var.

$P_G(x) = 1$

We will prove this by induction on $|E(G)|$.

2) $e \in E(G)$ ..... $e$ smyčka

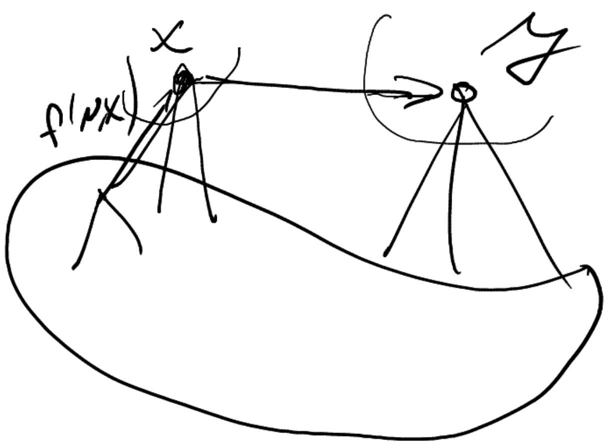

2) $P_G(x) = (x-1)P_{G-e}(x)$

3) $P_G(x) = P_{G/e}(x) - P_{G-e}(x)$

$P_G(x) = P_{G-e}(x) \cdot (x-1)$

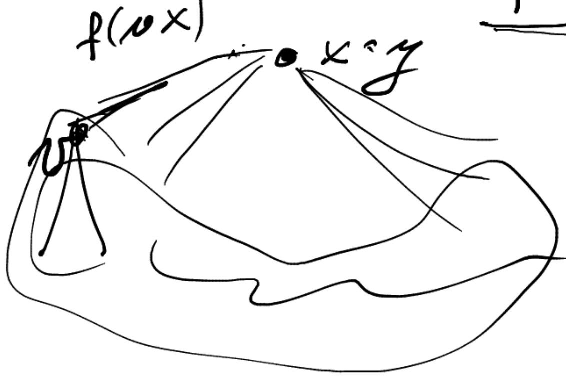
$x = |\Gamma|$

3) e není smyčka




$$\boxed{P_{G/e}(x) = P_G(x) + P_{G-e}(x)}$$ rekurze

f ... tok v G/e




G   f(xy) je def. 2×

... Kirch. z. v x a v y

... obojí stejně

... ☺

$\boxed{f \text{ je NZ v } G/e}$ ⟹ ∃ jednozn. tok $\tilde{f}$ v G

BUĎ $\tilde{f}$ je NZ v G

NEBO NE ... $\tilde{f}(e)=0$

... $\tilde{f}$ je NZ tok v G−e

## Tutte polynomial

Contraction/deletion invariant – a polynomial in two variables that counts NZ flows, colorings and many more graph invariants. The Tutte polynomial is usually denoted $T_G(x, y)$ and satisfies the relation $T_G = T_{G-e} + T_{G/e}$ if $e$ is neither a loop, nor a bridge, with the base case $T_G(x, y) = x^i y^j$ for $G$ with $i$ bridges, $j$ loops, and no other edges. One can use $T_G$ to express the flow polynomial $P_G$ as well as the *chromatic polynomial* $C(x)$ (the number of proper colorings using $x$ colors).

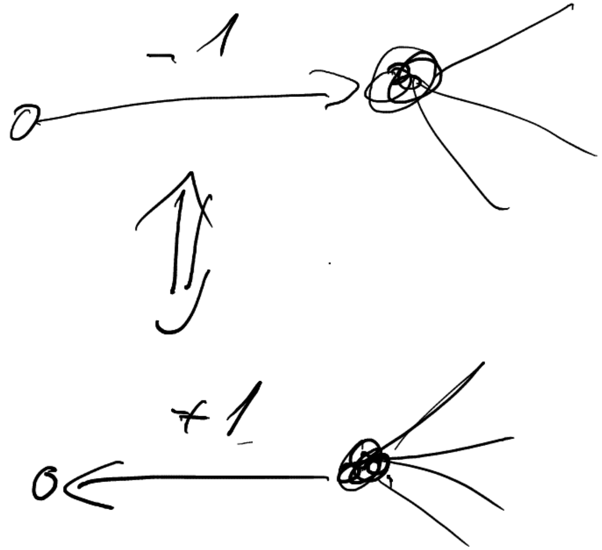
## Reversing orientations

We need directed edges for the definition of flows. However, we will in fact study undirected graphs. To understand why, let us define a simple notation. Let $G$ be a digraph, $f$ a mapping $E(G) \to \Gamma$ and $F \subseteq E(G)$ any set of edges. We let $G_F$ denote the digraph obtained from $G$ after reorienting all edges in $F$. We define a mapping $f_F$ as follows:

$$f_F(e) = \begin{cases} -f(e) & \text{if } e \in F \\ f(e) & \text{otherwise} \end{cases}$$

**Observation 7.** *Let $f$ be a $\Gamma$-flow on a digraph $G$, let $F \subseteq E(G)$. Then $f_F$ is a $\Gamma$-flow on $G_F$. Moreover, if $f$ is NZ then $f_F$ is also NZ.*

We can consider all pairs $(G_F, f_F)$ to be different representations of "the same flow" and we pick the most convenient one.



## Easy properties of flows

The following easy observation connects $\mathbb{Z}_2$-flows with cycles ($\neq$ circuits).

**Observation 8** ($\mathbb{Z}_2$-flow)**.** *Let $G$ be a graph and $f$ any $\mathbb{Z}_2$-flow on $G$. Then the* support *of $f$ (that is, the set of edges with nonzero value of $f$) is a cycle.*

*In particular a graph has a NZ $\mathbb{Z}_2$-flow iff it is a cycle.*

**Theorem 9** ($\mathbb{Z}_3$-flow of cubic graphs)**.** *Let $G$ be a cubic (i.e., 3-regular) graph. Then $G$ admits a NZ $\mathbb{Z}_3$-flow iff $G$ is bipartite.*

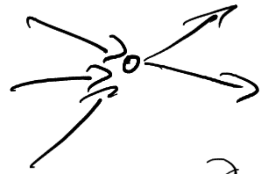
*Proof.* If $G$ is bipartite, we direct all edges from one part to the other and assign 1 to each edge, clearly this is the desired flow. On the other hand, . . . □

$0 = 1+1+1$



$\operatorname{supp} f = \{e : f(e) \neq 0\}$

$f$ je NZ . . . $\operatorname{supp} f = E(G)$

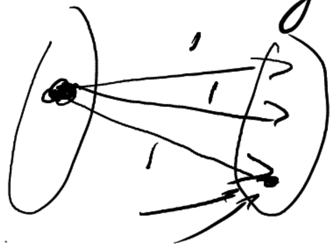


$f^+(v) = f^-(v)$

$v\ \mathbb{Z}_2$ : $\underbrace{f^+(v) + f^-(v)}_{\text{sudý počet !}} = 0$

$v \operatorname{supp} f$ má každý vrchol sudý stupeň

$\Rightarrow$ jde rozložit na kružnice




$1+1+1 = 0$

G je kubický a $f$ .... NZ $\mathbb{Z}_3$-tok $\longrightarrow$ $f(e) \in \{+1, -1\}$

búno $\forall e$ $f(e) = +1$

$\forall v$ $\quad f^+(v) = f^-(v)$ $\longrightarrow$ partity : $\{v : \deg^+(v) = 0\}$

$0 = 1+1+1$ $\qquad\qquad$ $\{v : \deg^-(v) = 0\}$

$1+1+1 = 0$

$\cancel{1+1 = 1}$

žádná hrana v rámci partity $\square$

**Theorem 10** ($\mathbb{Z}_2^2$-flow of cubic graphs)**.** *Let $G$ be a cubic (i.e., 3-regular) graph. Then $G$ admits a NZ $\mathbb{Z}_2^2$-flow iff $G$ is edge 3-colorable.*

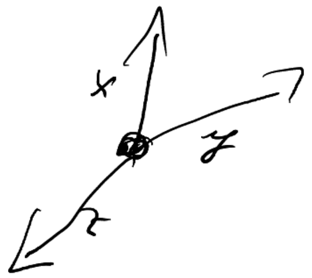
$\Rightarrow$ ✓ $\quad \Leftarrow$

- As opposed to the previous two characterisations (being a cycle and being bipartite), the condition in this theorem is NP-complete to check.
- We will frequently meet graphs that are cubic and *fail to have edge 3-coloring* $\Rightarrow$ *snarks.*

*Proof.* • A NZ $\mathbb{Z}_2^2$-flow can use only values from $A = \{(0,1),(1,0),(1,1)\}$. (NE $(0,0)$)

- As we are calculating modulo 2, we don't care about the orientation.
- It is easy to check that if three elements of $A$ sum to zero, they must be in fact distinct.

... ☐




$x, y, z \in A$

$x + y + z = 0$

kdyby $x = y$ ..... $x + y = 0$

$z = 0$ ↯

$x, y, z$ navzájem různé

Tj.: NZ $\mathbb{Z}_2^2$-tok je hranové obarvení barvami $(0,1)$, $(1,0)$, $(1,1)$.

Naopak ještě lehčí.

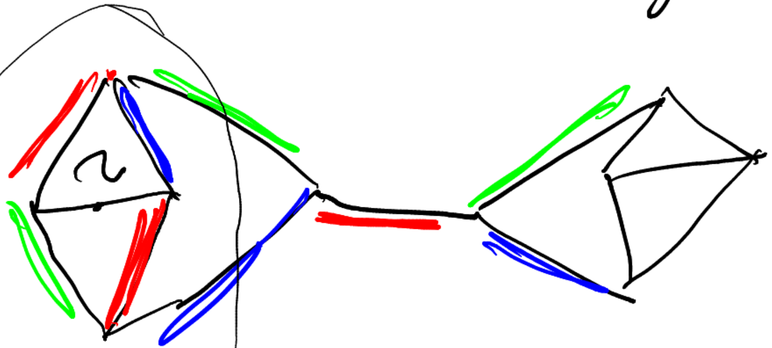
**Corollary 11** (3-edge-coloring and bridges)**.** *Let $G$ be a cubic graph with at least one bridge. Then $G$ is not edge-3-colorable.*

(hr. 3-ob. $\Longleftrightarrow$ NZ $\mathbb{Z}_2^2$ tokem — nejde pro most)

In analogy with the chromatic number $\boxed{\chi(G)}$ we define the flow number of a graph $G$ to be

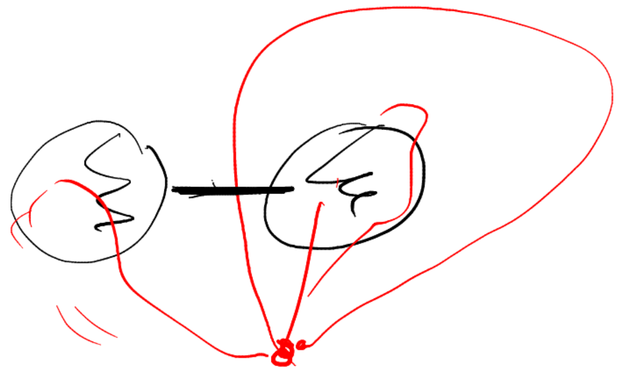
$$\boxed{\varphi(G)} = \inf\{|\Gamma| : G \text{ has a NZ } \Gamma\text{-flow}\}\,;$$

- $\varphi(G)$ is defined (as $\infty$) if $G$ has no NZ flow.
- This happens iff $G$ has a bridge.
- (In analogy: what graphs have no proper coloring?)
- Monotonicity: compare with $\chi$.



cv. (3

smyčka



G má NZ $\mathbb{Z}_3$-tok

$\Downarrow$ ?

G má NZ $\mathbb{Z}_2$-tok ?

($\mathbb{Z}_4$-tok ?)

stejné

**Definition 12.** *Let $G$ be a digraph, $f$ a $\mathbb{Z}$-flow on $G$.*
*$f$ is a $k$-*flow *if $|f(e)| < k$ $(\forall e)$.*
*$f$ is a* nowhere-zero $k$-flow *if $0 < |f(e)| < k$ $(\forall e)$.*

$\pm 1, \pm 2, \dots, \pm(k-1)$

*$k$-NZF := nowhere-zero $k$-flow*
*$\Gamma$-NZF := nowhere-zero $\Gamma$-flow*

Note: Many authors use $k$-flow to mean NZ $k$-flow.

**Theorem 13** (Tutte)**.** *A graph has a $k$-NZF iff it has $\mathbb{Z}_k$-NZF.*

$\Longrightarrow$ lehké

$f$ ... $k$-NZF ... $f(e) \in$

... $f$ splňuje kirch. zák. v $\mathbb{Z}$

$f'$ ... $f$ mod $k$ ... zobr. $E(G) \to \mathbb{Z}_k$

$\Longleftarrow$ těžší, později ... $f'$ splňuje k.z. v $\mathbb{Z}_k$

$f(e) \neq 0$

Motivated by this result we will sometimes use *$k$-flow* to mean $\Gamma$-flow for any $\Gamma$ of size $k$.

**Corollary 14** (group-monotonicity). *Let* $\Gamma_1$, $\Gamma_2$ *be groups, with* $|\Gamma_1| \leqslant |\Gamma_2|$. *Then any graph with* $\Gamma_1$*-NZF has also a* $\Gamma_2$*-NZF.*

$|\Gamma_1| = k_1$

$|\Gamma_2| = k_2$

minulá věta

Dk $\Gamma_1$-NZF $\Longleftrightarrow$ $\mathbb{Z}_{k_1}$-NZF $\Longleftrightarrow$ $k_1$-NZF

tok. pol.

$\Downarrow$ ---- zřejmé

$\Gamma_2$-NZF $\Longleftrightarrow$ $\mathbb{Z}_{k_2}$-NZF $\Longleftarrow$ $k_2$-NZF

PŘÍŠTĚ: NZ TOKY V ROVINNÉM GRAFU G

$\Updownarrow$

OBARVENÍ $G^*$