2.9.2 Rationality

Ironically, $\sqrt{2}$ can sometimes make things rational:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2. \tag{2.33}$$

Hence, by the *principle of the excluded middle*,

$$\text{either } \sqrt{2}^{\sqrt{2}} \in \mathbf{Q} \text{ or } \sqrt{2}^{\sqrt{2}} \notin \mathbf{Q}.$$

In either case, we can deduce that there are irrational numbers α and β with α^β rational. But how do we know which ones? This is not an adequate proof for an Intuitionist nor a Constructivist. It is entirely ineffective, and we may build a whole mathematical philosophy project around such issues. Actually, $\sqrt{2}^{\sqrt{2}}$ is transcendental by the Gelfond-Schneider theorem (Exercise 27), but proofs of this are hard and usually suffer from the same flaws.

It is instructive to compare this result with the assertion that $\alpha = \sqrt{2}$ and $\beta = 2\log_2(3)$ yield $\alpha^\beta = 3$ as *Mathematica* confirms. This illustrates nicely that verification is often easier than discovery. Similarly, the fact that multiplication is easier than factorization is at the base of secure encryption schemes for e-commerce.

Indeed, there are eight possible rational/irrational triples: $\alpha^\beta = \gamma$; finding examples of all cases is now a good exercise (Exercise 28). Note how much can be taught about computation with rational numbers, approximation to irrationals, rates of convergence, etc. from these simple pieces.

We close the section with a description of the meeting between the inventor of logarithms (John Napier) and the scientist who made them into technology (Henry Briggs) [279]:

> [W]here almost one quarter hour was spent, each beholding the other with admiration before one word was spoken: at last Mr. Briggs began "My Lord, I have undertaken this long journey purposely to see your person, and to know by what wit or ingenuity you first came to think of this most excellent help unto Astronomy, viz. the Logarithms: but my Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy."

2.10 Commentary and Additional Examples

1. **The hardest possible proof?** Use Fermat's last theorem to prove $2^{1/n}$ is irrational for integer $n > 2$. Generalize.

2. **The final digit of a sum.** Problem: Determine the final digit ℓ_n of $\sigma_n = \sum_{k=1}^n k$. (Taken from [158]).

   Solution: Computational experimentation shows the pattern repeats modulo 20; $\ell_{n+20} = \ell_n \mod 20$ is easily proven from $\sigma_{n+20} = \sigma_n + \sum_{n+1}^{n+20} k$.

3. **The $3x+1$ problem.** This is a classic example of an innocent looking, but highly intractable problem:

   > For integer x, let $T(x) = (3x+1)/2$ for x odd and $x/2$ for x even. The $3x+1$ conjecture is that starting from any positive integer n, repeated iteration of T eventually returns to 1.

   This problem is best described in the interactive article by Jeff Lagarias at http://www.cecm.sfu.ca/organics/papers/lagarias (see also [198]), with records stored at http://www.ieeta.pt/~tos/3x+1.html. This conjecture has been "checked" to at least $100 \cdot 2^{50}$.

4. **Limit of a simple iteration.** Establish the limit of the iteration that starts with $a_0 = 0, a_1 = 1/2$ and iterates $a_{n+1} = (1 + a_n + a_{n-1}^3)/3$, for $n > 1$. Determine what happens as $a_1 = a$ is allowed to vary.

5. **Putnam problem 1985–B5.** Evaluate

   $$\mathcal{K} = \int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\, dt.$$

   Answer: $\mathcal{K} = \sqrt{\pi}e^{-3970}/\sqrt{1985}$. The Putnam problems listed here and in subsequent chapters are taken from [185]. *Hint*: This is problematic to evaluate numerically as stated, since its value is tiny. So consider instead $\mathcal{K}(\alpha) = \int_0^\infty t^{-1/2} e^{-\alpha(t+t^{-1})}\, dt$ for some other specific (or general) constant α.

6. **Putnam problem 1987–A6.** Let n be a positive integer and let $a_3(n)$ be the number of zeroes in the ternary expansion of n. Determine for which positive x the series $\sum_{n=1}^\infty x^{a_3(n)}/n^3$ converges. Answer: For $x < 25$. In the b-ary analogue, $\sum_{n=1}^\infty x^{a_b(n)}/n^b$ converges if and only if $x < b^b - b + 1$.

7. **Putnam problem 1987–B1.** Evaluate

   $$\int_2^4 \frac{\sqrt{\log(9-x)}}{\sqrt{\log(9-x)} + \sqrt{\log(3+x)}}\, dx \qquad (= 1).$$

8. **Putnam problem 1991–A5.** Find the supremum of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx,$$

for $0 \le y \le 1$. *Hint*: Plot the function.

9. **Putnam problem 1992–A2.** Evaluate

$$\int_0^1 C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k}\, dy,$$

where $C(\alpha)$ is the coefficient of x^{1992} in the Maclaurin expansion of $(1+x)^\alpha$. Answer: 1992.

10. **Putnam problem 1992–B3.** Consider the dynamical system generated by $a_0 = x$ and

$$a_{n+1} = \frac{y^2 + a_n^2}{2},$$

for $n \ge 0$. Determine the region in the plane for which the iteration converges. What is its area? *Hint*: Try computing some values and plot the results. Assuming without loss of generality $x, y > 0$, the limit must satisfy $2\ell - \ell^2 = y^2$. Thus the region defined is the convex hull of unit circles centered at $(\pm 1, 0)$. Answer: $\pi + 4$.

11. **Random projections.** Consider a arbitrary point inside a triangle. Determine what happens asymptotically when the point is projected to successive sides of the triangle, where the side is selected either in cyclical order or pseudo-randomly. *Hint*: Consider first what happens in an obtuse triangle.

12. **Putnam problem 1995–B4.** Determine a simple expression for

$$\sigma = \sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}}}. \tag{2.34}$$

Hint: Calculate this limit to 15 decimal place accuracy, using ordinary double-precision arithmetic. Then use the ISC tool, with the "integer relation algorithm" option, to recognize the constant as a simple algebraic number. The result can be proved by noting that $\sigma^8 = 2207 - 1/\sigma^8$, so that $\sigma^4 + \sigma^{-4} = 47$. Answer: $(3+\sqrt{5})/2$.

13. **Berkeley problem 1.1.35.** Find the derivative at $x = 0$ of

$$\int_{\sin(x)}^{\cos(x)} e^{t^2 + xt}\, dt.$$

Hint: Plot it. Answer $= (e-3)/2$. The Berkeley problems listed here and in subsequent chapters are taken from [128].

14. **Berkeley problem 7.6.6.** Compute A^{10^6} and A^{-7} for

$$A = \begin{bmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

Hint: Use *Mathematica* or *Maple* to evaluate A^n for various integers n.

15. **Two radical expressions.** (From [167, pg. 81, 84]). Express

$$\sqrt[3]{\cos\left(\frac{2}{7}\pi\right)} + \sqrt[3]{\cos\left(\frac{4}{7}\pi\right)} + \sqrt[3]{\cos\left(\frac{6}{7}\pi\right)}$$

$$\sqrt[3]{\cos\left(\frac{2}{9}\pi\right)} + \sqrt[3]{\cos\left(\frac{4}{9}\pi\right)} + \sqrt[3]{\cos\left(\frac{8}{9}\pi\right)}$$

as radicals. *Hint*: Calculate to high precision, then use the ISC tool to find the polynomial they satisfy.

Answers: $\sqrt[3]{\frac{1}{2}(5 - 3\sqrt[3]{7})}$ and $\sqrt[3]{\frac{3}{2}\sqrt[3]{9} - 3/2}$.

16. **Some simple continued fractions.** Compute the simple continued fraction for $e, (e-1)/2, e^2, \log(2), \log(10), 3^{1/2}, 2^{1/3}, \pi, \pi/2, e^\pi$, and π^e.

17. **Crandall's continued fraction.** Compute, then guess and prove, the continued fraction for

$$\sqrt{2}\frac{e^{\sqrt{2}} + 1}{e^{\sqrt{2}} - 1},$$

which fraction is manifestly *not* periodic, proving in this way that $e^{\sqrt{2}}$ is irrational. Richard Crandall informs us that this (in 1968) was the first and perhaps only interesting thing he ever proved all by himself.

18. **Putnam problem 1988–B2.** Prove or disprove: If x and y are real numbers with $y \ge 0$ and $y(y+1) \le (x+1)^2$, then $y(y-1) \le x^2$. *Hint*: Plot it.

19. **Putnam problem 1989–A3.** Show that all roots of

$$11\,z^{10} + 10\,iz^9 + 10\,iz - 11 = 0$$

lie on the unit circle. *Hint*: This can be solved explicitly using *Maple*, *Mathematica*, or a custom-written root-finding program that employs Newton iterations. A detailed discussion of polynomial root-finding techniques can be found in Section 7.3 of the second volume.

20. **Putnam problem 1992–A3.** For a given positive integer m find all triples (n, x, y) of positive integers with $(n, m) = 1$ solving $(x^2 + y^2)^m = (xy)^n$. *Hint*: Using a symbolic math program, try finding solutions for various integer pairs x, y. Answer: The only solution is $(m+1, 2^{m/2}, 2^{m/2})$ for m even.

21. **Berkeley problem 6.11.5.** Prove that $\sqrt{2} + \sqrt[3]{3}$ is irrational. *Hint*: Use *Maple* or *Mathematica* to find the minimum polynomial of this constant.

22. **The happy end problem.** The happy end problem is to find, for $n \geq 3$, the smallest positive integer $N(n)$ such that any set of $N(n)$ points, no three of which are collinear, must contain n points that are the vertices of a convex n-gon. It is so called because Ester Klein, who posed the problem, married George Szekeres shortly after he and Paul Erdős proved the first bounds on the problem [205]. It is still open [205].

23. **The Mann iteration.** For any continuous function $f : [0,1] \mapsto [0,1]$, the iteration $x_0 = x \in [0,1]$ and

$$x_n = \frac{1}{n} \sum_{k=0}^{n-1} f(x_k)$$

(the Césaro average) always converges to a fixed point of f. One can study many other summability methods similarly. Since the function can be highly obstreperous, this is largely a theoretical real-variable iteration, albeit a beautiful one. Easy examples will convince one of how painfully slow or unstable convergence can be. This is especially understandable in light of

Theorem 2.1 (Sharkovsky). *If a continuous self-mapping of the reals, f, possesses a periodic point of order m, then f will possess a periodic point of order n, precisely when n follows m in the following ordering of the natural numbers:*

$$3, 5, 7, \cdots 2{\cdot}3, 2{\cdot}5, 2{\cdot}7, \cdots 2^2 3, 2^2 5, 2^2 7, \cdots 2^3 3, 2^3 5, 2^3 7, \cdots, \cdots, 2^3, 2^2 2, 2, 1.$$

In particular this includes the famous result of Li and Yorke to the effect that "period three implies chaos."

24. **Finding coefficients of an integer polynomial.** The numerical identification methods described here work especially well when you know *a priori* that a given integral, sum or other constant is a linear combination of given quantities, and you wish only to obtain the precise integer coefficients. For example, if you are told that, for any integer $N > 0$,

$$\sum_{n=0}^{\infty} \frac{n^n e^{-n}}{(n+N)!}$$

evaluates to a polynomial of degree N in e, then it is an easy matter, for any given small N, to pick off the integers. Indeed, for $N < 10$, say, you can discover that

$$Q_N \;=\; \sum_{n=0}^{\infty} \frac{n^n e^{-n}}{(n+N)!} - \sum_{k=1}^{N} \frac{(-1)^{k-1} e^k}{(N-k)!\,k^k} \qquad (2.35)$$

is a rational number. The first four are $-1, -1/4, -7/108, -97/6912$. Continuing in this manner, you can ultimately discover that

$$Q_N \;=\; \sum_{k=1}^{N} \frac{(-1)^k}{(N-k)!k^k} \sum_{n=0}^{k-1} \frac{k^n}{n!}. \qquad (2.36)$$

This relies on replacing $\exp(-n)$ by its series, exchanging order of summation, and then discovering and deriving the identity

$$\sum_{k=0}^{n} (-1)^k k^n \binom{N+n}{N+k} = (-1)^n \sum_{k=1}^{N} k^n (-1)^{k-1} \binom{N+n}{N-k} \qquad (2.37)$$

or equivalently,

$$\sum_{k=0}^{M+N} (-1)^k (M-k)^M \binom{M+N}{k} \;=\; 0 \qquad (2.38)$$

for all $M, N > 0$. This in turn follows (using the binomial theorem) from

$$\sum_{k=0}^{P} (-1)^k k^n \binom{P}{k} \;=\; 0, \qquad (2.39)$$