Probabilistic Methods

Some Useful Formulas
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(This is based on a tex-file I found somewhere on internet and forgot where – thanks to the author anyway!)

1. Estimates of factorials

   (a) (Stirling's Formula):
   $$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right)$$
   Most often, this is too precise – and also the fact that it is not a bound is slightly inconvenient. So we may use weaker inequalities:

   (b) $n! \le n^n$

   (c) $e\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n$

   (d) $\left(\frac{n}{e}\right)^n \le n! \le \left(\frac{n+1}{2}\right)^n$

2. Let $n \ge k \ge 0$, then we have:

   (a) $\binom{n}{k} \le 2^n$

   (b) $\binom{n}{k} \le \frac{n^k}{k!}$

   (c) $\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{ne}{k}\right)^k$.

3. (middle binomial coefficient)

   (a) $\frac{2^{2n}}{2\sqrt{n}} \le \binom{2n}{n} \le \frac{2^{2n}}{\sqrt{2n}}$

   (b) More precisely,
   $$\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}}\cdot(1+o(1))$$

4. $\binom{n}{\alpha n} = 2^{n(H(\alpha)+o(1))}$

5. (estimating $1+t$)

   (a) (from above) For all $t \in R$ we have $1+t \le e^t$ with equality holding only at $t = 0$.

   (b) (from below) For all small $t \in R$ we have $1+t \ge e^{ct}$ for appropriate c. E.g., if $p \in [0, 1/2]$ we have $1-p \ge e^{-2p}$.

6. For all $t, r \in R$, such that $n \ge 1$ and $|t| \le n$. (ed: here is a typo – what is the correct form?)
   $$e^t\left(1-\frac{t^2}{r}\right) \le \left(1+\frac{t}{r}\right)^n \le e^t.$$

7. For all $t, r \in R^+$,
$$\left(1+\frac{t}{r}\right)^r \leq e^t \leq \left(1+\frac{t}{r}\right)^{r+t/2}.$$

8. For any $n \in N$, the n-th Harmonic number is
$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} = \ln n + \Theta(1).$$

9. Let X be sum of independent Poisson trials, and $E[X] = \mu$.
   - For $\delta > 0$ we have, $\mathbf{Pr}[X > (1+\delta)\mu] < \left[\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right]^\mu$.
   - For $0 < \delta \leq 1$ we have, $\mathbf{Pr}[X < (1-\delta)\mu] < e^{-\mu\delta^2/2}$.

10. Let S_n denote the distribution
$$S_n = X_1 + X_2 + \ldots + X_n$$
where $\mathbf{Pr}(X_i = 1) = \mathbf{Pr}(X_i = -1) = 1/2$ and the X_i are mutually independent. Then $\mathbf{Pr}(S_n > \lambda\sqrt{n}) < e^{-\lambda^2/2}$ for all $n, \lambda \geq 0$.

11. Let $C_u(G)$ denote the expected length of a walk that starts at u and ends upon visiting every vertex in G at least once. The cover time of G, denoted $C(G)$, is defined by $C(G) = \max C_u(G)$. Then $C(G) \leq 2m(n-1)$.

12. Let $c = X_0, X_1, \ldots, X_n$ be a martingale sequence such that for each $i \leq n-1$,
$$|X_{i+1} - X_i| \leq 1$$
Then
$$\mathbf{Pr}(|X_n - c| > \lambda\sqrt{n}) < 2e^{-\lambda^2/2}.$$

13. Suppose that $G(x)$ is the generating function of a probability distribution $p_0, p_1, \ldots$. Then we have
   - $\mathbf{Pr}(X \leq r) \leq x^{-r}G(x)$ for $0 < x \leq 1$.
   - $\mathbf{Pr}(X \geq r) \leq x^r G(x)$ for $1 \leq x$.