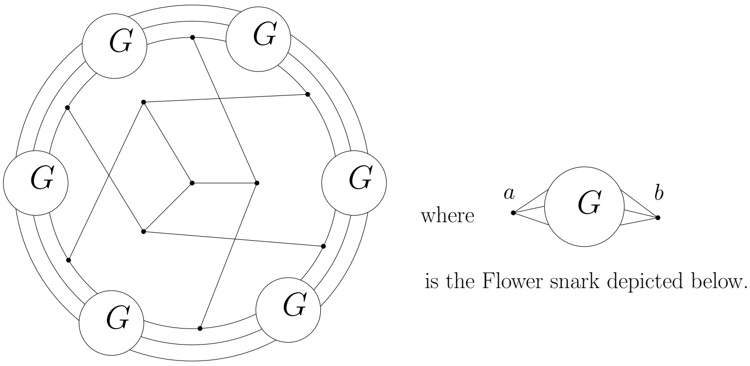
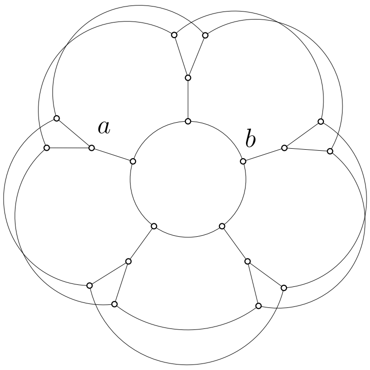
Flows and cycles in graphs – Exercises 9

1. In class we did prove that the graph in the figure is a snark. Prove that it is cyclically 6-edge-connected: that is, when we delete less than 6 edges from the graph, at most one component will contain a cycle. (For a partial solution try with 4 or 5 instead of 6.)





2. Let G be a digraph (graph with directed edges) and Γ a *ring* (that is, we can add and multiply, we have unity and all the usual properties except division). Let $\mathcal{F}$ be the set of all Γ-flows (not necessarily nowhere-zero) on G, considered as a vector space, subspace of $\Gamma^{E(G)}$. Similarly, let $\mathcal{T}$ be the vector space of all Γ-tensions on G. Prove that $\mathcal{F}$ is orthogonal complement of $\mathcal{T}$ and vice versa, i.e., $\mathcal{T} = \mathcal{F}^\perp$ and $\mathcal{F} = \mathcal{T}^\perp$.

3. Suppose that G, H are digraphs and $h : V(G) \to V(H)$ be a homomorphism (that is, for every edge (u, v) of G, there is an edge $(h(u), h(v))$ in H). Let Γ be an abelian group. Show that there is a Γ-tension continuous map from G to H: that is a map $f : E(G) \to E(H)$ such that for every Γ-tension t on H, the composition tf is a Γ-tension on G.

4. Prove that for a cubic graph G the following are equivalent:

1. G has a cycle-continuous mapping to the Petersen graph
2. G has a Petersen coloring
3. G has a normal coloring

We recall the definitions: A mapping $f : E(G) \to E(H)$ is *cycle-continuous* (another word for $\mathbb{Z}_2$-flow-continuous) if for every cycle $C \subseteq E(H)$ the preimage $f^{(-1)}(C)$ is a cycle in G.

Petersen coloring of a cubic graph G is a mapping $f : E(G) \to E(Pt)$ such that whenever a, b, c are three distinct edges of G sharing a vertex, its images $f(a)$, $f(b)$, $f(c)$ are three distinct edges of Pt sharing a vertex.

Finally, a *normal coloring* of a cubic graph is a coloring of the edges by 5 colors that is (a) a proper coloring (no two adjacent edges have the same color) and (b) every edge is either rich or poor. Here an edge e is called *rich* if among the four edges adjacent to e, four colors are used; edge is called *poor* if only two colors are used there.