(1) $M = \{(x,y,z);\ xy \geq 2,\ x+y+z \leq 3,\ x,y,z \geq 0\}$

$V(M) = \iiint_M dx\,dy\,dz$, M je uzavřená a měřitelná (∂M má míru 0)

meze pro integraci – F.v.

$0 \leq z \leq 3-x-y$

(pro $z=0$) $y \leq 3-x$

a ($x>0$) $\frac{2}{x} \leq y$

a tedy: $\frac{2}{x} \leq 3-x$

pro $x^2-3x+2 \leq 0$, tj. $(x-2)(x-1) \leq 0$ $\quad x \in \langle 1,2\rangle$

Tedy, $V(M) = \int_1^2 dx \int_{\frac{2}{x}}^{3-x} dy \int_0^{3-x-y} dz =$

$$= \int_1^2 dx \int_{\frac{2}{x}}^{3-x} (3-x-y)\,dy = \int_1^2 \left[(3-x)y - \frac{y^2}{2}\right]_{\frac{2}{x}}^{3-x} dx =$$

$$= \int_1^2 \left((3-x)^2 - \frac{(3-x)^2}{2} - (3-x)\cdot\frac{2}{x} + \frac{2}{x^2}\right) dx$$

$$= \int_1^2 \left(\frac{(3-x)^2}{2} - \frac{6}{x} + 2 + \frac{2}{x^2}\right) dx =$$

$$\left[-\frac{(3-x)^3}{6} - 6\ln x + 2x - \frac{2}{x}\right]_1^2 =$$

$$= -\frac{1}{6} - 6\ln 2 + 4 - 1 - \left(-\frac{8}{6} + 2 - 2\right) =$$

$$= \frac{7}{6} + 3 - 6\ln 2 = \underline{\underline{\frac{25}{6} - 6\ln 2}}$$

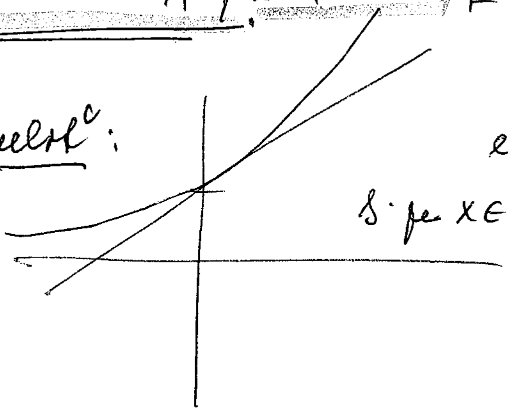
(2) $f_n(x) = e^{-nx}(1+x)^n, \quad x \in (-1,+\infty)$

a) ~~xta~~ <u>bodová limita:</u> $\quad x=0 \quad f_n(0) = 1 \to 1$

$$\lim_{n\to\infty} f_n = f(x) = \begin{cases} 0, & x \in (-1,0) \cup (0,+\infty) \\ 1, & x=1 \end{cases}$$

$x \neq 0 \quad \lim_{n\to\infty} e^{-nx}(1+x)^n = \lim_{n\to\infty} \left(\frac{1+x}{e^x}\right)^n = 0$

<u>neboť:</u>



$e^x \geq 1+x$

tj. pro $x \in (-1,+\infty) - \{0\}$ je $0 < \frac{e^x}{1+x} < 1 \Rightarrow$

$\Rightarrow \lim_{n\to\infty} \left(\frac{e^x}{1+x}\right)^n = 0$

b) <u>stejnoměrná limita</u> $\quad f_n \not\rightrightarrows f$, neboť f_n spojité na $(-1,+\infty)$ a f nespojitá v $x=0$

<u>neboť:</u> $\sup_{x\in(-1,+\infty)} |f_n(x) - f(x)| = \sup_{x\in(-1,0)} |e^{-nx}(1+x)^n| \quad (f_n(0) - f(0) = 0)$

$= \sup_{x\in(-1,+\infty)} |f_n(x)| :$

f_n

$f_n'(x) = -n e^{-nx}(1+x)^n + e^{-nx} \cdot n(1+x)^{n-1} = e^{-nx} n (1+x)^{n-1}\left(-(1+x)+1\right)$

$= -e^{-nx} \cdot n \cdot x (1+x)^{n-1}$

na $(-1,+\infty)$ je $f_n'(x) = 0 \Leftrightarrow x=0$, v $(-1,0)$ $f_n \nearrow$ $(f_n' > 0)$, na $(0,+\infty)$ $f_n \searrow$ $(f_n' < 0)$

$\Rightarrow \sup_{x\in(-1,+\infty)} f_n(x) = \max_{x\in(-1,+\infty)} f_n(x) = f_n(0) = 1,$

tj. $\lim_{n\to\infty} \sup_{x\in(-1,+\infty)} |f_n(x) - f(x)| = 1 \not\to 0$

c) lokálně stejnoměrná konvergence

$x \in (-1, \cancel{+a} \rangle$, ~~okolí~~ : $-1 < a < 0$ $\qquad (f_n \nearrow)$ na $(-1,0)$

$$\sup_{x \in (-1,-\varepsilon\rangle} |f_n(x) - f(x)| = f_n(-\varepsilon) = e^{-n\varepsilon}(1 \cancel{+\varepsilon})^n \Rightarrow 0$$

$x \in \langle a, +\infty)$, $\quad 0 < a \quad \ldots (f_n \searrow)$ na $(0,+\infty)$

$$\sup_{x \in \langle a,+\infty)} |f_n(x) - f(x)| = |f_n(a)| = e^{-na}(1+a)^n \to 0$$

per $n \to \infty$

$\Rightarrow f_n \rightrightarrows 0$ na $(-1, a\rangle$, $-1 < a < 0$ a $\langle a, +\infty)$ per $0 < a$ $\Big\} \Rightarrow f_n \overset{loc}{\rightrightarrows}$ na $(-1,0)$ a $(0,+\infty)$

ale ~~není~~ $\not\overset{loc}{\rightrightarrows}$ na $(-1,+\infty)$

(nelimnujeme stejnoměrné na žádném okolí 0 !)

Hodnocení: konvergence 4b + F. řada (def.) (−1b)
a_k 9b — 4 — rozložit 1b, 0 + int. 4b, dosazení 4b
b_k 5b rozložit 1b, int. p.p. 2, dosazení 2b

(3) $g(x) = \cos\frac{x}{2} + x$, $x \in (-\pi, \pi)$ ... Fourierova řada Φ_g

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1) je-li $\tilde{g}$ 2π-per. rozšíření ($\tilde{g}(\pi)$ lib.),

pak $\tilde{g}(\pi-) = g(\pi-) = \pi$, $\tilde{g}(\pi+) = g(-\pi+) = -\pi$

g je po částech hladká, spojitá v $(-\pi,\pi)$, tj.

$$\boxed{\Phi_g(x) = \begin{cases} g(x) & v\ (-\pi,\pi) \quad (\text{resp. } \tilde{g}(x) \text{ pro } x \in ((2k-1)\pi, (2k+1)\pi),\ k\in\mathbb{Z}) \\ 0 & \text{pro } x = (2k+1)\pi,\ k \in \mathbb{Z} \end{cases}}$$

$$\left(\Phi_g(x) = \frac{\tilde{g}(x+) + \tilde{g}(x-)}{2} \quad v\ \mathbb{R}\right)$$

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2) $\Phi_g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\cos\frac{x}{2} + x\right)dx = \frac{2}{\pi}\int_0^{\pi}\cos\frac{x}{2}\,dx = \frac{2}{\pi}\left[\sin\frac{x}{2}\right]_0^{\pi}\cdot 2 = \frac{4}{\pi}$$

$$a_k = \frac{1}{\pi}\Big(\int_{-\pi}^{\pi}\cos\frac{x}{2}\cos kx\,dx + \underbrace{\int_{-\pi}^{\pi} x\cos kx\,dx}_{=0\ (\text{lichá fce})}\Big) = \frac{2}{\pi}\int_0^{\pi}\cos\frac{x}{2}\cos kx\,dx =$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\left(\cos\frac{2k+1}{2}\cdot x + \cos\frac{2k-1}{2}\cdot x\right)dx =$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\left[\frac{\sin\frac{2k+1}{2}x}{\frac{2k+1}{2}} + \frac{\sin\frac{(2k-1)}{2}\cdot x}{\frac{2k-1}{2}}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\sin\left(\frac{\pi}{2}+k\pi\right)}{2k+1} + \frac{\sin\left(-\frac{\pi}{2}+k\pi\right)}{2k-1}\right)$$

$$= \frac{2}{\pi}\left(\frac{(-1)^k}{2k+1} + \frac{(-1)^{k+1}}{2k-1}\right) = \frac{2(-1)^k}{\pi}\cdot\frac{\overbrace{2k-1-(2k+1)}^{-2}}{4k^2-1} =$$

$$\text{(také } (-\ \text{''}\ + \frac{(-1)^{k-1}}{(2k-1)}) \quad\Bigg|\quad = \frac{4}{\pi}\cdot\frac{(-1)^{k+1}}{4k^2-1}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\cdot\sin kx + x\sin kx\right)dx = \frac{1}{\pi}\Big(\underbrace{\int_{-\pi}^{\pi}\cos\frac{x}{2}\cdot\sin kx\,dx}_{=0} + \underbrace{\int_{-\pi}^{\pi}x\sin kx\,dx}_{\text{sudá fce}}\Big)$$

$$= \frac{2}{\pi}\int_{0}^{\pi}x\sin kx\,dx \underset{\text{pp.}}{=} \left|\begin{matrix} u'=\sin kx, & u=\frac{-\cos kx}{k} \\ v=x & , v'=1\end{matrix}\right| \text{(lichá fce)}$$

$$= \frac{2}{\pi}\left(\left[-\frac{\cos kx}{k}\cdot x\right]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos kx\,dx\right) =$$

$$= \frac{2}{\pi}\Big(-\frac{\pi}{k}\cos k\pi + \frac{1}{k}\underbrace{\left[\frac{\sin kx}{k}\right]_0^{\pi}}_{=0}\Big) = \frac{2}{k}\cdot(-1)^{k+1}$$

5. $$\Phi_g(x) = \frac{2}{\pi} + 2\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{2}{\pi(4k^2-1)}\cdot\cos kx + \frac{1}{k}\sin kx\right)$$

$$\left(= \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{4\cdot(-1)^{k+1}}{\pi(4k^2-1)}\cos kx + \frac{2\cdot(-1)^{k+1}}{k}\sin kx\right)$$