!WARNING!

This is very preliminary version, likely to contain tons of errors, some of them may by fatal!

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1 Introduction

Throughout this text, *digraph* will denote directed multigraph, that is a graph with directed edges, loops and multiple edges allowed. If G is a digraph, $V(G)$ denotes its set of vertices, $E(G)$ its set of edges. By writing $e = uv$ (or $e = (u, v)$) we mean that e is *one of possibly many* edges starting at u and ending at v.

Let us also agree that *group* will always refer to an abelian group, that is one with commutative operation.

The following is the principal notion of this text.

Definition 1. *Let G be a digraph, Γ a group. A mapping $f : E(G) \to \Gamma$ is called a* flow *(or, more explicitly, a Γ-flow), if for every vertex $v \in V(G)$ the Kirchhoff law is valid:*

$$\sum_{e=(v,u)} f(e) = \sum_{e=(u,v)} f(e)\,.$$

We will frequently use $f^+(v)$ to denote the left-hand side of the above equation and $f^-(v)$ to denote the right-hand side of the above equation. So $f^+(v)$ denotes the amount of flow that leaves v, and $f^-(v)$ the amount of flow that enters v.

??? example ???

It is immediate that $f \equiv 0$ is a flow. Also if f, g are flows, then $f \pm g$ are also flows, thus the set of all Γ-flows on a given digraph is again an (abelian) group. If Γ is actually a field, than the set of all flows is even a vector space. We will discuss later what is its dimension and how to find a basis.

Before going further with our development, it's time for more notation. Whenever A, B are sets of vertices, we let $f(A, B)$ be the sum of $f(e)$ over all edges e starting in A and ending in B. Also we let $f^+(A) = f(A, \bar{A})$ and $f^-(A) = f(\bar{A}, A)$ (where $\bar{A} = V(G) \setminus A$).

Observation 2. *Let G be a digraph, Γ a group, f a Γ-flow. Then for any set $A \subseteq V(G)$ we have*

$$f^+(A) = f^-(A)\,.$$

Proof. Let us sum the Kirchhoff law for all $v \in A$. It is easy to check that

$$\sum_{v\in A} f^+(v) = f^+(A) + f(A, A)$$

and

$$\sum_{v\in A} f^-(v) = f^-(A) + f(A, A)$$

which finishes the proof. □

Corollary 3 (flow and small cuts)**.** *Let G be a digraph, Γ a group, f a Γ-flow.*

- *If e is a bridge then $f(e) = 0$.*
- *If e, e' form a 2-cut (and a oriented in the direction) then $f(e) + f(e') = 0$.*

Corollary 4 (flow and a partition)**.** *Let G be a digraph, Γ a group, f a Γ-flow. Consider any partition $\mathcal{P}$ of $V(G)$. Let $G_{\mathcal{P}}$ be the graph where each equivalence class is identified to a vertex and all edges are preserved. Then f is a Γ-flow on $G_{\mathcal{P}}$.*

So the set of all flows is easy to understand, let's try something harder. (Actually, there is a better reason to pursue in this direction, as will be later revealed by the duality with coloring.)

Definition 5. *Let G be a digraph, Γ a group, f a Γ-flow. We say that f is a* nowhere-zero Γ-flow*, if $f(e) \neq 0$ for all edges $e \in E(G)$. Frequently we will shorten nowhere-zero to NZ.*

It's easy to see that a graph with a bridge has no nowhere-zero flow. We will see later, that the opposite is also true, that is a graph with no bridge has nowhere-zero flows in every sufficiently large group. Before that, however, we will try to understand what is the dependence on the group Γ.

Theorem 6 (flow polynomial)**.** *[Tutte 1954] For every graph G there is a polynomial $P_G(x)$ (called the* flow-polynomial *of G) such that for every group Γ the number of nowhere-zero Γ-flows on G is $P_G(|\Gamma| - 1)$.*

Notes: There are two surprising informations in one statement. One is that the number of NZ Γ-flows only depends on the size of Γ, the other that this dependence is given by a polynomial. Note that for counting all Γ-flows (not only NZ ones) the result is also true. This may be observed by going trough the same proof, but it is in fact easy to obtain using elementary cycles, as we will see later.

Proof. We will prove this by induction over $|E(G)|$. Graph with no edges has exactly one nowhere-zero flow (which one is it?). Next, suppose that G has an edge, say e, and that for all graphs with smaller number of edges the theorem is true. We will distinguish two cases.

e is a loop: then erasing e has no effect on the remaining graph. To make this precise, f restricted to $E(G) \setminus \{e\}$ is a NZ flow on $G - e$ and each NZ flow on $G - e$ can be extended to exactly $|\Gamma| - 1$ NZ flows on G. Thus in this case we simply have

$$P_G(x) = x P_{G-e}(x) .$$

e is not a loop: now we will contract e. Again, $f|_{E(G)\setminus\{e\}}$ is a NZ flow on G/e. Suppose g is a NZ flow on G/e. There is a unique way how to extend it to a flow g' on G: we assign to e such value that will satisfy the Kirchhoff law for one end of e and check that it is also satisfied at the other. Flow g', however, may not be NZ, as we may be forced to have $g'(e) = 0$. If this happens, however, g is a NZ flow not only on G/e but also on $G - e$. To wrap it up, the number of NZ flows on G/e equals the number of NZ flows on G plus the same on $G - e$. Consequently, we have

$$P_G(x) = P_{G/e}(x) - P_{G-e}(x) .$$

□

Notes: 1. If G contains a bridge e then of course $P_G(x) = 0$. We can still use the above equation to see that in such case $P_{G/e}(x) = P_{G-e}(x)$.

2. The two displayed equations above are example of so-called contraction/deletion invariants. A general polynomial of this type is the Tutte polynomial, a polynomial in two variables, that counts NZ flows, colorings and many more graph invariants. The Tutte polynomial is usually denoted $T_G(x, y)$. When we put $x = 0$, we get the

number of NZ flows in a group of size $y+1$ (that is our flow polynomial $P_G(y)$), when we put $y=0$, we get the number of proper colorings using x colors (so-called chromatic polynomial).

3. We need graphs with directed edges for the definition of flows to make sense. However, most problems in the area are in fact concerned with undirected graphs. To understand why, let us define a simple notation. Let G be a digraph, f a mapping $E(G) \to \Gamma$ and $F \subseteq E(G)$ any set of edges. We let G_F denote the digraph obtained from G after reorienting all edges in F. We define a mapping f_F as follows:

$$f_F(e) = \begin{cases} -f(e) & \text{if } e \in F \\ f(e) & \text{otherwise} \end{cases}$$

Observation 7. *Let f is a Γ-flow on a digraph G, let $F \subseteq E(G)$. Then f_F is a Γ-flow on G_F. Moreover, if f is NZ then f_F is also NZ.*

The proof is immediate, as in each of the Kirchhoff laws some terms are put on the other side with the opposite sign. However, this simple observation is extremely convenient. We can consider all pairs (G_F, f_F) to be different representations of "the same flow" and we can pick among these the most convenient one.

Exercises: **1.** Let G be a digraph, Γ a group, $f : E(G) \to \Gamma$ any mapping. Let $v_0 \in V(G)$ be any vertex and suppose that Kirchhoff law is satisfied for all vertices in $V(G) \setminus \{v_0\}$, then f is a flow.

2.

(a) Find flow polynomial for K_4. Apply for groups $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$.

(b) Without using part (a) determine the number of NZ $\mathbb{Z}_4$-flows in K_4. (With any orientation.)

(c) Without using parts (a), (b) determine the number of NZ $\mathbb{Z}_2^2$-flows in K_4.

3. Find a NZ Γ-flow for the Petersen graph using as small group Γ as possible.

2 Easy properties of flows

We start by a simple definition to unify terminology. A *circuit* is a 2-regular connected graph. A *cycle* is a graph with all degrees even (not necessarily connected). Sometimes *the edge-set* of such graph is referred to as a cycle as well.

The following easy observation connects $\mathbb{Z}_2$-flows with cycles.

Observation 8 ($\mathbb{Z}_2$-flow)**.** *Let G be a graph and f any $\mathbb{Z}_2$-flow on G. Then the* support *of f (that is, the set of edges with nonzero value of f) form a cycle.*

In particular a graph has a NZ $\mathbb{Z}_2$-flow iff it is a cycle.

Theorem 9 ($\mathbb{Z}_3$-flow of cubic graphs)**.** *Let G be a cubic (i.e., 3-regular) graph. Then G admits a NZ $\mathbb{Z}_3$-flow iff G is bipartite.*

Proof. If G is bipartite, we direct all edges from one part to the other and assign 1 to each edge, clearly this is the desired flow. On the other hand, if G has a NZ $\mathbb{Z}_3$-flow, we may (reorienting G if necessary) assume that all edges assume the value 1. Clearly, if G is oriented in this way, then all vertices have all edges outgoing or all incoming. This defines the desired bipartition. $\square$

Theorem 10 ($\mathbb{Z}_2^2$-flow of cubic graphs)**.** *Let G be a cubic (i.e., 3-regular) graph. Then G admits a NZ $\mathbb{Z}_2^2$-flow iff G is edge-3-colorable.*

Notes: As opposed to the previous two characteristics (being a cycle and being bipartite), the condition in this theorem is NP-complete to check. We will meet frequently graphs that a cubic and *fail to have edge 3-coloring*. Such graphs (with some further assumptions, such as absence of bridges) are called *snarks* and will be discussed later on.

Proof. A NZ $\mathbb{Z}_2^2$-flow can use only values from $A = \{(0,1),(1,0),(1,1)\}$. As we are calculating modulo 2, we don't care about the orientation. It is easy to check that if three (not necessarily distinct) elements of A sum to zero, they must be in fact distinct. Clearly if they are distinct, they sum to zero. Thus each NZ $\mathbb{Z}_2^2$-flow is in fact a proper edge-coloring using elements of A. □

Corollary 11 (3-edge-coloring and bridges)**.** *Let G be a cubic graph with bridges. Then G is not edge-3-colorable.*

Proof. By the previous theorem, if G were edge-3-colorable, G admits a NZ $\mathbb{Z}_2^2$-flow, which is impossible if G has a bridge. □

In analogy with the chromatic number we define the flow number of a graph G to be

$$\varphi(G) = \inf\{|\Gamma| : G \text{ has a NZ } \Gamma\text{-flow}\}\,;$$

the infimum ensures that the $\varphi(G)$ is defined (as ∞) if G has no NZ flow. As we will see soon, this happens iff G has a bridge.

From previous results we know, that if Γ is *any* group with $\varphi(G)$ elements, then G has a NZ Γ-flow. What if Γ is larger? It may seem that we have more choices when seeking for the NZ flow, but it's not quite clear. However, the following definition (and theorem) will imply positive answer.

Definition 12. *Let G be a digraph, f a $\mathbb{Z}$-flow on G.*
We say that f is a k-flow *if $|f(e)| < k$ for each edge e.*
We say that f is a nowhere-zero k-flow *if $0 < |f(e)| < k$ for each edge e.*
From now on, we will shorten nowhere-zero k-flow as k-NZF and nowhere-zero Γ-flow as Γ-NZF and

Notes: Many authors use k-flow to mean NZ k-flow.

Theorem 13 (Tutte)**.** *A graph has a k-NZF iff it has $\mathbb{Z}_k$-NZF.*

(Proof is postponed.)

Notes: Motivated by this result we will sometimes use *k-flow* to mean Γ-flow for any Γ of size k.

Corollary 14 (group-monotonicity)**.** *Let Γ_1, Γ_2 be groups, with $|\Gamma_1| < |\Gamma_2|$. Then any graph with Γ_1-NZF has also a Γ_2-NZF.*

Proof. Let $k_i = |\Gamma_i|$. If G has Γ_1-NZF, then is has a $\mathbb{Z}_{k_1}$-NZF, hence a k_1-NZF. This is by definition also a k_2-NZF, hence G has a $\mathbb{Z}_{k_2}$-NZF and a Γ_2-NZF. □

NZ flows in planar graphs A general way to construct NZ flows comes from colorings and planar duality. We now present just a sample to show one of the early motivations for the study of NZ flows.

Let G be a planar digraph, consider a proper coloring of faces of G by elements of some group Γ – so that faces sharing an edge get distinct colors. Now for an edge e let $f(e)$ be the difference of the value of the left face and of the right face. It's easy to check that f is a NZ Γ-flow. This in fact works for graphs drawn on arbitrary surface. For planar graphs, however, we will see that all NZ flows arise in this way, leading to equality $\varphi(G) = \chi(G^*)$. This will be proved later. Now we will only notice that this implies $\varphi(G) \le 4$ whenever G is planar. We saw already, that $\varphi(\mathrm{Pt}) = 5$ (where Pt is the Petersen graph). Perhaps surprisingly, it is open, whether $\varphi(G)$ can be larger than 5.

Theorem 15 (Jaeger)**.** *The following are equivalent for any graph G*

1. *G has a $\mathbb{Z}_2^2$-NZF*
2. *$E(G)$ is a union of two cycles*

Proof. Let f is a NZ $\mathbb{Z}_2^2$-flow on G, observe it only uses values $(0,1)$, $(1,0)$, $(1,1)$. We let f_1 denote the first coordinate, f_2 the second one. Clearly, f_i is a $\mathbb{Z}_2$-flow, hence its support is a cycle. Moreover, each edge e is in at least one of these cycles, as otherwise $f(e) = (0,0)$, a contradiction.

In the other direction we can proceed in the same way: if $E(G) = E_1 \cup E_2$ and each E_i is a cycle, then there is a $\mathbb{Z}_2$-flow f_i that is 1 precisely on E_i. Putting $f = (f_1, f_2)$ we get the desired flow.

An alternative proof is to consider 2-flows g_i on E_i and then observe that $g = 2g_1 + g_2$ is a NZ 4-flow (which by the above results implies existence of a NZ $\mathbb{Z}_2^2$-flow. □

Exercises: **4.** Find NZ 5-flow for the Petersen graph. Find NZ 4-flow for K_4.

5. Prove that cubic graph with a bridge has no edge-3-colorings, without resorting to NZ flows.

6. When proving that the Petersen graph does *not* have some property (in the previous set of exercises we discussed edge 3-coloring, resp. NZ $\mathbb{Z}_2^2$-flow) it is helpful, that the graph is extremely symmetric. Proving these symmetries is the topic of this exercise. First few ad-hoc definitions:

We say that graph G is *H-transitive*, if whenever H_1, H_2 are subgraphs of G, both isomorphic to H, there is an automorphism of G which maps H_1 to H_2.

We say that graph G is *ordered H-transitive*, if whenever H_1, H_2 are subgraphs of G, both isomorphic to H, and $f : H_1 \to H_2$ is an isomorphism, then there is an automorphism of G which extends f.

(a) *Kneser graph $K(n,k)$* is a graph which has k-subsets of an n-set as vertices, and two vertices are adjacent iff the corresponding sets are disjoint. Show that the Petersen graph is isomophics with $K(5,2)$.

(b) The Petersen graph is K_1-transitive (or vertex-transitive).

(c) The Petersen graph is K_2-transitive (or edge-transitive).

(d) The Petersen graph is ordered K_2-transitive (or arc-transitive).

(e) The Petersen graph is ordered T-transitive where T is a tree with 5 edges, which in the Petersen graph is formed by an edge and the four edges adjacent to it.

(f) The Petersen graph is M-transitive, where M is a matching with 5 edges.

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3 Intermezzo – musing about disjoint spanning trees

For this part large part of the audience was missing – so the topic is such, that it can be safely skipped, the main result will be stated again, when it will be needed. To capture the more leisurely pace of this class, this section is written as a dialogue. The idea is that two mathematicians, A and B, are trying to discover the result on their own (although A seems to know something in advance). Frequently, some argument/picture/calculation is omitted, to let the reader take part in the discussion.

A: What does a graph need to have a spanning tree?
B: It's enough to be connected.

A: What about two (edge-) disjoint spanning trees?
B: What about 2-connected?

A: Let's try some graph – a circuit perhaps?
B: Umm.

A: Let's try higher-connected graphs. Say, a cube.
B: Well, none of the trees can contain a vertex of degree 3, so both must be hamiltonian paths. And … cube has no such disjoint hamiltonian paths.

A: True. Can you say it simpler? (Counting proofs are nice!)
B: I see, the cube does not have enough edges!

A: Can you generalize it?
B: I suppose a graph with k disjoint spanning trees and n vertices must have at least $k(n-1)$ edges.

A: Is this enough?
B: No, it still needs to be connected. Wait, it is even k-edge-connected.

A: And this is sufficient?
B: I don't know …

A: It's not. Now when you know, find a counterexample :-)
B: Well … If I started with a graph that does not have k disjoint spanning trees (like the cube for $k = 2$), I could start adding edges somehow …

A: Sounds right. But you better not add them among the old vertices, or it may start to have 2 DST (disj.spanning trees).
B: I see. What about attaching a large clique at a vertex?

A: Yes, that's it. Can you do it without a cut-vertex?
B: If I let the cube and the clique have two vertices in common, it will have 2 DST …

A: Think about this: if U is a subset of $V(G)$, and you identify all vertices of U to a single vertex (preserving multiplicities). How to spanning trees of G look in G/U.
B: They are spanning trees again!

A: Careful!
B: … except they may contain a cycle …

A: And are they spanning?
B: Ummm, yes, they are. I see … If $G[U]$ has lots of edges, we may think that G has 2 DST. But G/U can still be a cube, so the 2 DST of G would become two disjoint spanning subgraphs of G/U, the cube …

A: Can this be generalized?

B: Perhaps we can contract more than one set?

A: Yes! The right notion is a partition. If a partition $\mathcal{P}$ consists of disjoint sets $V_1, \ldots, V_t$ that cover $V(G)$, then $G/\mathcal{P}$ means G where vertices of each set V_i were identified to a single vertex, preserving all the edges between distinct sets.

B: I see, and again, a spanning tree of G becomes a spanning tree, of $G/\mathcal{P}$. I mean spanning connected graph.

A: Precisely. What does this say about G with k DST?

B: For every partition $\mathcal{P}$ the graph $G/\mathcal{P}$ has also k DST. In particular $G/\mathcal{P}$ must be k-edge-connected and it has enough edges.

A: Let's stick with counting edges. What exactly did we get?

B: OK. $|E(G/\mathcal{P})| \geq k(|V(G/\mathcal{P})| - 1)$.

A: Let's try some special cases. What if $\mathcal{P}$ is the partition into 1-vertex parts?

B: I see, we get our original bound on the total number of edges.

A: Try the other extreme!

B: If $\mathcal{P}$ has just one class – it's not very illuminating ... I see, if $\mathcal{P}$ has two classes, we get that ... G has no cut of size $< k$!

A: So the condition with $G/\mathcal{P}$ seems rather strong ... And believe it or not, it actually characterizes graphs with k DST!

B: Awesome! We actually proved half of the equivalence already :-). Shall we try the other one?

A: Sure! :-) What do you suggest?

B: We may try induction on k. Or we may try to take some small cut, find DST in both parts and connect them somehow. Or ...

(few hours later :-))

B: Or we may try something else ...

A: Let me suggest an approach. Instead of looking for k DST, let us go for k edge-disjoint spanning forests, lets call them $F_1, \ldots, F_k$.

B: Sure, just take each F_i to be edge-less. So what?

A: And now try to put in the forests as many edges as possible. Say, do maximize $\sum |E(F_i)|$.

B: Sounds interesting. If all edges of the graph are used, then either we have k DST, or the graph has too few edges. So there must be some edges missing.

A: What can you say about such an edge?

B: It creates a cycle in every F_i.

A: Can you used the cycles to move around?

B: What do you mean?

A: It might be useful if there are many maximal k-tuples of forests.

B: I see ...

B: Yes, now I really see. We can pick some F_i, add the new edge and remove some other edge of the created cycle. This creates another maximal tuple.

A: Can you go any further?

B: Well, after we remove the other edge of the cycle, we cannot add it back to any other forest, F_j, as we would create a cycle. And I suppose we can go on and on ...

A: Will it ever stop?

B: It must, the graph is finite. But I still can't see what is it good for.

A: Try to draw a picture!

B: Umm, can it be that we get some part of G that contains k DST?

A: Yep! And this means we are done, aren't we?

B: But we wanted DST in the whole graph.

A: That's what induction is for ...

Theorem 16 (Nash-Williams, Tutte)**.** *A multigraph G contains k edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$ the contracted graph $G/\mathcal{P}$ satisfies*

$$|E(G/\mathcal{P})| \geq k(|V(G/\mathcal{P})| - 1) .$$

Exercises: **7.** Based on the previous dialogue, write down a complete (but short :-)) proof of the theorem.

8. Using the theorem, show that every $2k$-connected graph contains k edge-disjoint spanning trees.

9. In analogy with the above guess the characterization of graphs G, such that $E(G)$ can be decomposed into k forests. If you are brave enough, you can prove it along the same lines.

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4 Further basic properties ...

4.1 Postponed proof

Theorem 17 (Tutte)**.** *A graph has a k-NZF iff it has $\mathbb{Z}_k$-NZF.*

Proof. The forward implication is obvious. For the other one, let g be a $\mathbb{Z}_k$-NZF in a graph G. For any mapping $f : E(G) \to \mathbb{Z}$ we let $f(v)$ be the net flow out of a vertex v, that is $f(v) = \sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e)$. Recall that f is a flow iff $f(v) = 0$ for every vertex v. We won't achieve this directly, however, but by certain optimization.

Let $f : E(G) \to \mathbb{Z}$ be such that

1. $f(e) \equiv g(e) \pmod k$ for each edge e,
2. $|f(e)| < k$ for each edge e, and
3. subject to the above, $\sum_{v \in V(G)} |f(v)|$ is as small as possible.

(If the sum in part 3. is zero, then f is a flow and we are done.)

By possibly reorienting the edges of G we may assume that $f(e) > 0$ for each edge e.

Easily $f(v) \equiv 0 \pmod k$ for each vertex $v \in V(G)$. Let V^+, V^0, and V^- denote the vertices v for which $f(v)$ is positive, zero, and negative, respectively. If

$V^0 = V$ we are done. Otherwise, observe that both V^+ and V^- are nonempty (as $\sum_v f(v) = 0$) and pick $a \in V^+$.

Observe now, that either there is a directed $a - b$ path for some $b \in V^-$ or there is a set A containing a but not b such that no directed edge leaves A.

The second possibility immediately yields a contradiction:

$$\sum_{v \in A} f(v) = - \sum_{e \in \delta^- A} f(e) < 0$$

on the other hand the sum on the left-hand side contains non-negative terms, at least one of which is positive.

So there is a directed $a - b$ path P with $a \in V^+$, $b \in V^-$. We define a mapping f' by letting $f'(e) = f(e) - k$ for $e \in E(P)$, and $f'(e) = f(e)$ otherwise. Now it is easy to check that conditions 1. and 2. above are still satisfied, and $f'(v) = f(v)$ for all vertices $v \neq a, b$, while $f'(a) = f(a) - k$, $f'(b) = f(b) + k$. As $f(a) \geq k$ and $f(b) \leq -k$, it follows that f' is better than f in terms of 3., a contradiction. $\square$

4.2 Flows and spanning trees

In the following we will consider how spanning trees can be used to construct flows.

A. Sum Let T be a spanning tree of G. Now for every edge $t \in E(G) \setminus E(T)$ and every $a \in \Gamma$ we let $\varphi_{t,a}$ be the (unique) flow in G such that

- $\varphi_{t,a}(t) = a$
- $\varphi_{t,a}(e) = 0$ for $e \neq t$ and $e \in E(G) \setminus E(T)$

(We just assing $\pm a$ on the path in T connecting ends of t.) Such flow is called an *elementary flow with respect to T*.

Let now $\mathcal{F}_\Gamma(G)$ denote the vector space of all flows (here we need Γ to be a field, otherwise it is not a vector space). It is easy to see (Exercise!) that for any fixed spanning tree T the elementary flows $\{\varphi_{t,1} : t \in E(G) \setminus E(T)\}$ form a basis of $\mathcal{F}_\Gamma(G)$.

Note, however, that this provides no means of control on the edges of T, thus we can't use this to construct a NZ flow. Indeed, we didn't even need the graph to be bridgeless!

B. Product

Theorem 18. *Any 4-edge connected graph admits a $\mathbb{Z}_2^2$-NZF.*

Proof. If G is 4-edge connected, then there are two disjoint spanning trees, T_1 and T_2 (Theorem **??**). Let f_i be the $\mathbb{Z}_2$-flow on G that equals 1 on all edges not in T_i. (Such flow exists by XXX.) Now put $f = (f_1, f_2)$. This is indeed a $\mathbb{Z}_2^2$-flow, and if $f(e) = 0$ for some edge e then e lies in both T_1 and T_2, a contradiction. $\square$

Exercises: **10.** Every bridgeless graph has a NZF (in a large enough group).

11. Prove that the elementary flows provide a basis of the vector space of flow.

12. What is the orthogonal complement to $\mathcal{F}$?

Theorem 19 (Jaeger)**.** *Any bridgeless graph admits a $\mathbb{Z}_2^3$-NZF.*

Proof. Suppose first that G is 3-edge connected, we will use spanning trees similarly as in the construction of a NZ 4-flow. We let G' be the (multi)graph obtained from G by adding to each edge a new one, parallel to it. It is easy to see that G' is 6-edge connected, thus it contains three edge-disjoint spanning trees, T_1', T_2', T_3'. If some T_i' contains one of the new edges, we replace it with the "original edge" of G. After doing this for all edges and all i we get spanning trees T_1, T_2, T_3 that are perhaps not edge-disjoint, but each edge is contained in at most two of them.

Now we proceed as in the proof of Theorem **??**. Let f_i be the $\mathbb{Z}_2$-flow on G that equals 1 on all edges not in T_i, and put $f = (f_1, f_2, f_3)$. This is indeed a $\mathbb{Z}_2^3$-flow, and if $f(e) = 0$ for some edge e then e lies in each of T_1, T_2, T_3, a contradiction.

So the theorem holds for all 3-edge-connected graphs. To prove it for all bridgeless graphs, suppose there is a counterexample and choose one with minimal number of edges, let it be denoted G. By the first part of the proof, G has a 2-edge-cut formed by some two edges – say these are e_1 and e_2, Put $G' = G/e_1$. By minimality of G, there is a NZ $\mathbb{Z}_3^2$-flow f in G'. It is easy to check that it can be extended to G by letting $f(e_2) = f(e_1)$.

XXX

□

Let us summarize what we know about small flows (and also what we don't know).

Prehled (toky jsou omezené, barevnost ne !!!): In the following table we consider only bridgeless graphs

- 1-flow impossible
- 2-flow exists precisely in cycles
- 3-flow for cubic graphs exists precisely in bipartite graphs
- 3-flow should exist in every 4-edge-connected graph by a conjecture of Tutte. However, even assuming 10^{10}-edge-connectivity is not known to suffice.
- 4-flow for cubic graph is the same as 3-edge-colorability. By a conjecture of Tutte, every bridgless graph, that does not have Petersen graph as a minor, admits a 4-flow. This was proved (for cubic graphs) by pro kubicke dokazali Robinson, Seymour and Thomas (unpublished) by reducing to four-color theorem.
- 5-flow should exist in every graph by a conjecture of Tutte
- 6-flow exist in every graph [Seymour]
- 8-flow exist in every graph [Jaeger]

We saw earlier that the existence of NZ k-flow can be stated in several equivalent ways (as integer flow using values in abs. value $1, \dots, k-1$, as NZ $\mathbb{Z}_k$-flow, or as NZ Γ-flow in any group of size k). In the following we give two more equivalent formulations, which allow for fractional relaxation of flows.

Theorem 20. *Let G be a digraph, let $0 < a \le b$ be integers. Then the following are equivalent.*

1. *There is a $\mathbb{Z}$-flow f on G such that $a \le f(e) \le b$ for each edge e of G.*
2. *There is a $\mathbb{R}$-flow f on G such that $a \le f(e) \le b$ for each edge e of G.*
3. *For each $U \subset V(G)$ we have $\frac{a}{b} \le \frac{|\delta^+(U)|}{|\delta^-(U)|} \le \frac{b}{a}$.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): take any set U. As "the net flow over each cut is zero", we have

$$\sum_{e\in\delta^+(U)} f(e) = \sum_{e\in\delta^-(U)} f(e)\,.$$

Now we use the lower bound for all terms on the left-hand side, and the upper bound on the right-hand side; this yields $a|\delta^(U)| \le b|\delta^-(U)$, which is one of the desired inequalities. For the other one we use the upper bound on the left-hand side.

(3) $\Rightarrow$ (1): We call a $\mathbb{Z}$-flow reasonable if $0 \le f(e) \le b$ for each edge e. Such flows exist, e.g. the zero flow. We will find reasonable flow that is optimal in the following sense:

- $m := \min\{f(e) : e \in E(G)\}$ is as large as possible;
- among flows with the same m we choose the one with as few edges attaining $f(e) = m$ as possible.

We claim that the optimal reasonable flow does in fact satisfy $f(e) \ge a$ for every edge, which would prove (1). For contradiction, suppose there is an edge $e_0 = u_0v_0$ for which $f(e_0) = m < a$. If there is a path P from v_0 to u_0 such that

- $f(e) < b$ for each edge e on P that is in the direction from v_0 to u_0, and
- $f(e) > m + 1$ for each edge e on P that is in the other direction,

that we can improve f. On the cycle $P + e_0$ we increase f by one for edges in one direction and decrease f by one for the other edges; this will decrease the number of edges for which $f(e) = m$ (e_0 will no longer be one), possibly it will also increase m. Thus, there is no such path. By a standard argument (as seen in **??**) it follows that there is a set U containing v_0 but not u_0 such that all edges e leaving U satisfy $f(e) = b$ and all edges $e \ne e_0$ entering U have $f(e) \le m + 1 \le a$. Consider now the net flow of f over the cut $\delta(U)$:

$$|\delta^+(U)| \cdot b = \sum_{e\in\delta^+(U)} f(e) = \sum_{e\in\delta^-(U)} f(e) < |\delta^-(U)| \cdot a\,,$$

a contradiction.

XXX

$\square$

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tensions – def: sum to zero over a circuit (phys. analogy) – equiv.: dif. of a potential – duals to flows – as im/ker of some matrix

---

Theorem 21. *The following is equivalent for a plane graph G:*

1. *G has NZ k-flow*
2. *G^* has NZ k-tension*
3. *G has proper face-coloring by k colors.*

* ukazal jsem priklady, kdy to na jinych plochach selze (cyklus na toru, ktery nema stenove 2-obarveni; K4 v projektivni rovine, ktera ma stenove 3-obarveni; Petersen v projektivni rovine, ktery nema stenove 5-obarveni), ale rekl jsem ze pro orientovatelnou plochu plati alespon jedna implikace

5 Cycle double cover

* vyslovil jsem CDC hypotezu (pod nazvem "hypoteza o dvojitem pokryti cykly"), definoval jsem k-CDC, OCDC, rekl jsem zesileni hypotezy na 5-OCDC

    * ukazal jsem ze 4-NT ¡=¿ 3-CDC ¡=¿ 4-CDC

Theorem 22. *The following is equivalent for a graph G.*

1. *G has a 4-NZF.*
2. *G has has a 3-CDC.*
3. *G has has a 4-CDC.*

Theorem 23. *Every graph with k-OCDC has a k-NZF. The opposite inequality holds only for $k \leq 4$, that is a graph with a k-NZF ($k \leq 4$) has a k-OCDC.*

* ukazal jsem ze CDC hypotezu staci dokazat pro kubicke grafy - udelal jsem to pomoci nahrazeni vrcholu stupne ¿= 4 kruznici, ale rekl jsem taky ze by to slo pres Fleischnerovo splitting lemma, to jsem vyslovil, rekl nastin dukazu a detaily jsem dal k rozmysleni

Berge-Fulkerson conjecture * rekl jsem Berge-Fulkersonovu hypotezu a par poznamek (plati v hranove 3-obarvitelnych grafech, plati pro Petersena)

* zminil jsem ze je otevrene jestli kazdy kubicky graf ma konstantni pocet perfektnich parovani, ktere pokryvaji hrany; pripadne konstantni pocet PP, ktere maji prazdny prunik

* ukazal jsem ze Berge-Fulkerson ¡=¿ kazdy graf ma 4-pokryti 6 cykly (pricemz uz driv jsme mluvili o tom ze z 8-NT plyne 4-pokryti 7 cykly)

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6 Matching polytope and applications

We will look at various sets of edges geometrically. That is, we consider $\mathbb{R}^{E(G)}$ as a euclidean vector space (which it is) and study various polytopes in it. For a set $M \subseteq E(G)$, the we define c_M – the *characteristic vector of M* – by $c_M(e) = 1$ if $e \in M$, and $c_M(e) = 0$ otherwise.

Definition 24. *The* matching polytope *of a (multi)graph G is defined by*

$$MP(G) = \operatorname{conv}\{c_M : M \textit{ is a matching in } G\} .$$

It is not hard to see, that all vectors c_M (for a matching M) are in fact vertices of MP. Note, that we consider non-perfect matchings too, so the zero vector is a vertex of every matching polytope.

For many application it is desirable to obtain description of the matching polytope as an intersection of halfspaces. XXXapplications will follow shortly, for an (original) application in combinatorial optimization consider the task to find a maximal matching in a graph with weighted edges. This is the same as solving a linear program over the matching polytope, and we can do this using ellipsoid method. (We only need to provide an efficient representation of the matching polytope, for details see XXX.)

For each $f \in MP(G)$ and $v \in V(G)$ we have $\sum_{e \in \delta(V)} f(e) \leq 1$ (we sum over all edges incident with one vertex), as this inequality holds for all vectors c_M. This, however does not describe $MP(G)$ completely (Exercise!).

Next, we observe that for each vertex set X of odd size, each matching uses at most $(|X|-1)/2$ edges induced by X. Consequently, for each such X we have inequality

$$\sum_{e \in E(G[X])} f(e) \le \frac{|X|-1}{2},$$

satisfied for each $f = c_M$ and so for each $f \in MP(G)$. This is already enough to describe the matching polytope.

REKNI ROVNOU OBE VERZE

Theorem 25 (Edmonds)**.** *For every graph G we have*

$$\begin{aligned} MP(G) = \{f \in \mathbb{R}^{E(G)}: \quad & \textstyle\sum_{e \in \delta(v)} f(e) \le 1 \text{ for each } v \in V(G) \\ & \textstyle\sum_{e \in E(G[X])} f(e) \le \frac{|X|-1}{2} \text{ for each } X \subseteq E(G) \text{ of odd size}\}. \end{aligned}$$

Theorem 26. *Let $PMP(G)$ be the polytope of perfect matchings, that is $PMP = \operatorname{conv}\{c_M : M$ is a perfect matching in $G\}$. Then*

$$\begin{aligned} PMP(G) = \{f \in \mathbb{R}^{E(G)}: \quad & \textstyle\sum_{e \in \delta(v)} f(e) = 1 \text{ for each } v \in V(G) \\ & \textstyle\sum_{e \in \delta(X)} f(e) \ge 1 \text{ for each } X \subseteq E(G) \text{ of odd size}\}. \end{aligned}$$

XXXdefine $f(S) = \sum_{e \in S} f(e)$ and use it !!!

We say a graph G is an *r-graph*, if G is r-regular, and for every odd set of vertices X the size of the edge-cut $\delta(X)$ is at least r. For example, a 3-graph is the same as a bridgeless cubic graph (Exercise!).

Application 1 Every r-graph has a uniform cover by perfect matchings. That is, there is a list of perfect matchings such that each edge is in the same number of them. (Easily, this number must be one third of them.)

Proof. Let G be the graph and let $f(e) = 1/r$ for each edge of G. We will show that f is in the perfect matching polytope $PMP(G)$. Obviously the sum around each vertex equals 1. Now for each odd set X the size of $\delta(X)$ is at least r, which gives the other condition □

Corollaries of Application 1 1) Every bridgeless cubic graph has a uniform cover by perfect matchings.

2) Every bridgeless cubic graph has a perfect matching. (This of course has easier proofs.) It also has a perfect matching using any given edge. (This, too, can be proved by an application of Tutte's theorem, but it's always good to have another way.)

3) Every bridgeless cubic graph has a perfect matching that contains no odd cut of size 3. Indeed, every matching that is a part of the uniform cover works.

Application 2 [Kaiser, Králˇ, Norine] Every bridgeless cubic graph G has perfect matchings M_1, M_2 such that $|M_1 \cup M_2| \ge \frac{3}{5}|E(G)|$.

Proof. First use Application 1, namely the third corollary: Let M be a perfect matching that contains no odd cut of size 3. Define $f(e) = 1/5$ for $e \in M$ and $f(e) = 2/5$ elsewhere.

We check that f is in PMP_G. The sum around each vertex is 1. If X is an odd-size vertex set, then $|\delta(X)|$ is odd, therefore either 3, or at least 5. In the latter case, $\sum_{e \in \delta(X)} f(e) \ge 5 \cdot \frac{1}{5} = 1$, which we need. In the former case, we know by the choice of M that exactly one of the edges in $\delta(X)$ is in M, therefore $\sum_{e\in\delta(X)} f(e) = \frac{1}{5} + \frac{2}{5} + \frac{2}{5} = 1$.

As f is in the perfect matching polytope, f is a convex combination of c_{M_i} for some perfect matchings M_i. Put $S = E(G) \setminus M$. By definition of f, we have $f(S) = \frac{2}{5}|S|$, hence $c_{M_i}(S) \geq \frac{2}{5}|S|$ for some M_i involved in the convex combination for f. Now $|M \cup M_i| = |E(G)| \cdot (\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{5}) = \frac{3}{5}|E(G)|$. □

The above may be generalized as follows. For a graph G define $m_i(G)$ to be the maximum fraction of edges that can be covered by a union of i perfect matchings – that is

$$m_i(G) := \max\{\frac{|M_1 \cup \cdots \cup M_i|}{|E(G)|} : M_i \text{ are perfect matchings}\}\}$$

So we found that $m_2(G) \geq 3/5$ for every 3-graph G, and this bound is attained for the Petersen graph. [KKN] did further find that $m_3(G) \geq ?/?$ for a 3-graph G. If Berge-Fulkerson conjecture is true, we have $m_5(G) = 1$.

Exercises: 13. Prove that a 3-graph is the same as a bridgeless cubic graph. **14.** Find upper and lower bounds for $m_3(G)$ when G is a cubic bridgeless graph.

Find some bounds on $m_i(G)$ for a general i, and use this to estimate number of perfect matchings needed to cover all edges of a graph G.

Now we give the postponed proof of Theorem 26.

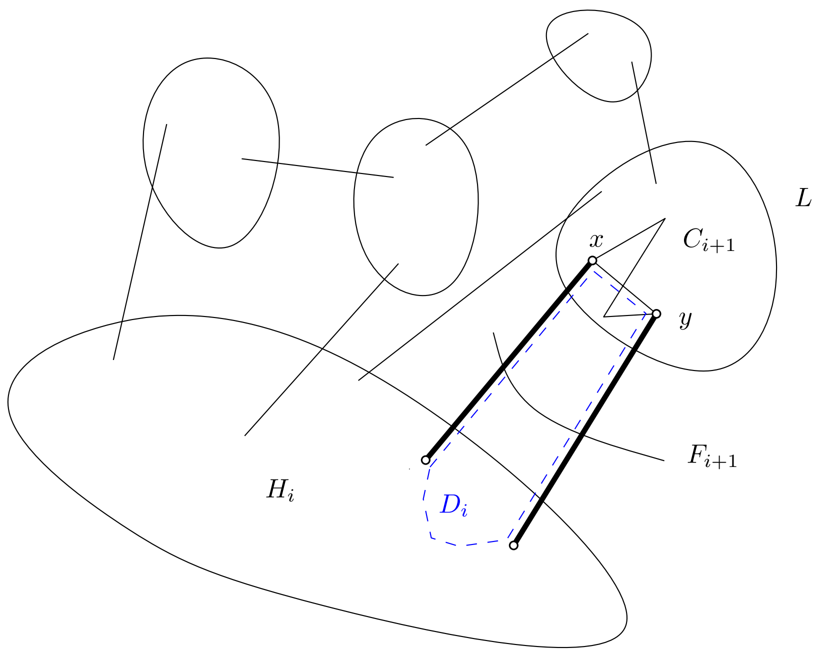
Proof. Let P_G be the polytope defined by the inequalities (**??**). Easily $PMP_G \subseteq P_G$, as all vertices of the perfect matching polytope (i.e., all c_M for M a parfect matching) satisfy the inequalities (**??**). For the other inclusion, we proceed by contradiction: we take the graph G with smallest $|V(G)| + |E(G)|$, and one vertex f of P_G such that $f \notin PMP_G$.

We have $0 < f(e) < 1$ for each edge e of G. If $f(e) = 0$ for some edge e, we let $G' = G - e$ and f' to be the restriction of f to $E(G')$. It is easy to check that $f' \in P_{G'}$, and as G' is smaller than G, we have $P_{G'} = PMP_{G'}$ and f' is a convex combination of characteristic vectors of perfect matchings of G'. When we take these matchings as perfect matchings of G (by extending the characteristic vector by a 0 in the coordinate indexed by e), we get $f \in PMP_G$, a contradiction.

On the other hand, if $f(e) = 1$ for some edge $e = uv$, then we put $G' = G - u - v$. Again, we let $f' = f|_{E(G')}$ and we check that $f' \in P_{G'} = PMP_{G'}$. By extending all the perfect matchings that occur in the convex combination for f' by the edge e we get perfect matchings whose convex combination is f, again a contradiction.

G has no vertices of degree ≤ 1. G certainly does not have isolated vertices (by inequality (**??**)), and if v is a vertex incident only with an edge e, then $f(e) = 1$, which we already disproved. Consequently, $|E(G)| \geq |V(G)|$.

Case 1. $|E(G)| = |V(G)|$ G is 2-regular, thus a disjoint union of circuits. None of these is odd (otherwise we let X be set of vertices of an odd circuit and get a contradiction with inequality (**??**)). For even circuits it is easy to ... (Exercise!).



Case 2. $|E(G)| > |V(G)|$ As f is a vertex of a polytope in $\mathbb{R}^{E(G)}$, at least $|E(G)|$ of the inequalities are satisfied with an equality. (Exercise!) Thus, one of them must be (*) $\sum_{e\in\delta(X)} f(e) = 1$ for some $X \subseteq V(G)$, such that $1 < |X| < |V(G)|$ and $|X|$ is odd. As $|X|$ is odd, every perfect matching of G contains an edge of $\delta(X)$. This together with (*) implies, that each of the matchings involved in the representation of f contain exactly one edge of $\delta(X)$. This suggest, we may want to treat X as a single vertex: if there is a representation for f, then this change of the graph will transform them in matchings.

To put this formally, we let $G_1 = G/X$ – all vertices of X are identified to a single vertex, we keep possible multiedges) – and $G_2 = G/\bar{X}$ (where $\bar{X} = V(G) \setminus X$). Again, let f_i be the restriction of f to the edge-set of G_i ($i = 1, 2$). It is easy to check that $f_i \in P_{G_i}$, which implies (Exercise!) that there are perfect matchings $(M_{i,k})_{k=1}^N$ of G_i such that

$$f_i = \frac{1}{N} \sum_{k=1}^{N} c_{M_{i,k}} . \tag{1}$$

Recall that each $M_{i,k}$ contains exactly one of the edges of $\delta(X)$ (we abuse the notation slightly, we identify the edges of $\delta(X)$ in G, and the corresponding edges of G_1, G_2). Moreover, if e is one of these edges, then the number of perfect matchings $M_{i,k}$ of G_i for which $e \in M_{i,k}$ is $Nf_i(e)$ (just look at the e-th coordinate of (1)). However, $Nf_1(e) = Nf_2(e) = Nf(e)$ (recall f_i was defined as a restriction of f to $E(G_i)$). Consequently, we may pair up the matchings of G_1 and of G_2 to agree on the edges of $\delta(X)$, indeed we may assume that $M_{1,k}$ and $M_{2,k}$ contain the same edge from the cut Z. We put $M_k = M_{1,k} \cup M_{2,k}$. It is easy to check that f is the average of c_{M_k}, which finishes the proof. □

Theorem 27 (Seymour)**.** *Every bridgeless graph G has a 6-NZF.*

Proof. Equivalently, we will show it has NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow. First, we can assume that G is 3-edge-connected (with the same proof as in the case of 8-NZF). We will find a spanning cycle C and carefully chosen edges between various components of

C. The plan is to use a $\mathbb{Z}_2$-flow f with support $E(C)$ and a $\mathbb{Z}_3$-flow g that is NZ outside of $E(C)$.

We will recursively define subgraphs $(H_i)_{i\ge 0}$ of G, cycles $(C_i)_{i\ge 1}$ and sets of edges $(F_i)_{\ge 1}$. To start, let H_0 be any vertex of G. If H_i is defined, we consider a decomposition of $G' = G - V(H_i)$ into 2-edge-connected components–blocks. (If $V(G) = V(H_i)$, we stop and put $n := i$.) The structure of this decomposition is such that after contracting each of the blocks, we obtain a forest. We take any leaf of this forest at let L be a block of G' corresponding to it.

We observe that $|\delta_{G'}(L)| = 1$ (by the choice of L), while $|\delta_G(L)| \ge 3$ (as G is 3-edge-connected). This implies there are at least two edges connecting L with H_i, we let F_{i+1} be the set of some two of them, and x, y be the ends of those edges in L. As L is 2-edge-connected, there are two edge-disjoint $x-y$ paths, and their union is a connected cycle, let it be denoted C_{i+1}. (If $x = y$, we may choose C_{i+1} to be empty.) We put $H_{i+1} = H_i + C_{i+1} + F_{i+1}$ (We do not add spanned edges.)

We let $C = \cup_{i=1}^n C_i$, $F = \cup_{i=1}^n F_i$, $H = H_n$. All edges of G are of three types: $E(C)$, F, and the rest, denoted by R. As claimed above, C is a spanning cycle, so it is easy to take a $\mathbb{Z}_2$-flow with support $E(C)$. We now define a $\mathbb{Z}_3$-flow that is non-zero on $R \cup F$

We observe (by induction on i) that all graphs H_i are connected, so we take a spanning tree $T \subseteq H$ and let g_n be a $\mathbb{Z}_3$-flow that equals 1 on $E(G) \setminus E(T)$. Next, we define $g_{n-1}, \dots, g_0$ so that each g_k is a $\mathbb{Z}_3$-flow that is nonzero on R and on F_j with $j > k$. If g_{i+1} is already defined, we consider a cycle D_i containing both edges of F_{i+1}, some $x-y$ path in C_{i+1} and any path in H_i that connects the other ends of the edges of F_{i+1}. Let φ_i be a $\mathbb{Z}_3$-flow that is nonzero on D_i. Consider flows $g_{i+1} + \alpha\varphi_i$ for $\alpha = 0, 1, 2$. At least one of them is nonzero on both edges of F_{i+1}, while we didn't change edges of R neither of $F_{>i+1}$. Consequently, the mapping $g = g_0$ is nonzero on $R \cup F$ and (f, g) is the desired $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF on G. $\square$

Notes: 1) Recall the standard proof of the fact that graph of maximum degree at most k is $(k+1)$-colorable. The second phase of the above proof is an analogue of this for $k = 2$. Indeed, if the graph G/C (each component of C is contracted to a vertex) is planar, then we are using the fact that the dual $(G/C)^*$ is 2-degenerate. As we saw, the argument works even for non-planar graphs. The nontrivial part is, of course, to find the cycle C such that G/C has this degenerate property.

2) It's tempting to try and use similar ideas to get a 5-flow conjecture. For this, one may say the above proof in an alternative way: we find a 2-flow f and a 3-flow g, that are not both equal to zero at the same edge. Then $2g + f$ is a NZ 6-flow. Now one may try to find a 2.5-flow g instead, that is a real-valued flow such that $1 \le g(e) \le 1.5$ for each edge e for which $f(e) = 0$. This would indeed produce a 5-flow. Exercise: discuss why does the above approach fail.

Exercises: 1) Describe PMP_G when G is a disjoint union of even circuits.

2) Let P be a polytope $\{x \in \mathbb{R}^d : Ax \le b\}$. Let V denote the vertices of P. Let x be a point of P. a) x is a convex combination of at most $d+1$ elements of V. b) If A and b have rational entries then x is a convex combination of some elements of V *with rational coefficients*. c) There is a list $v_1, \dots, v_n$ of vertices from V (possibly with repetition), such that $x = (v_1 + \cdots + v_n)/n$.

3)* Try to modify the proof of 6-NZF theorem to work for 5-NZF (as indicated in the notes below the proof). Describe what makes this approach fail. (If you succeed in proving the existence of 5-NZF, let humanity know! – see the list of open problems ...)

Let us first recall a useful lemma due to Fleischner.

Lemma 28. *Let G be a connected bridgeless graph, v a vertex with $\deg v \geq 4$, and e_0, e_1, e_2 three of its incident edges. Suppose that $G_{[v:e_0,e_1,e_2]}$ is connected. (This in particular holds whenever G is 2-connected.) Then at least one of $G_{[v:e_0,e_1]}$, $G_{[v:e_0,e_2]}$, is bridgeless connected.*

(Proof omitted.)

TODO: def $G_{[v:F]}$

In the last section we saw Seymour's proof of the existence of NZ 6-fllow. Tutte's 5-flow conjecture is still elusive, let us however look at some simple observations. In the following, G is a minimal counterexample to the conjecture. Explicitely: G is a bridgeless graph that admits no NZ 5-flow and among such graphs G has the smallest $|V(G)| + |E(G)|$.

(1) G **is 2-connected** Suppose not; then $G = G_1 \cup G_2$ where graphs G_1 and G_2 share just one vertex, and both are bridgeless. By minimality of G, both G_1 and G_2 admit a NZ 5-flow, thus G has it, too.

(2) G **is cubic** Suppose not, let v be a vertex such that $\deg v \neq 3$. If $\deg v = 1$, then G has a bridge, contradiction. If $\deg v = 2$, then we can suppress this vertex (contract one of its incident edges). The graph we obtain is smaller, so has a NZ 5-flow, which is easily extended back to G. Finally, let $\deg v \geq 4$, let (as in the Fleischner's lemma), e_0, e_1, e_2 be three of the incident edges. As G is 2-connected, the lemma implies that one of graphs $G_i = G_{[v:e_0 e_i]}$ $(i = 1, 2)$ is bridgeless. After suppressing the newly created vertex of degree 2, we get a graph G'_i that has the same number of vertices as G but one edge less – thus it admits a NZ 5-flow f_i. It is easy to extend it back to G_i and then to G, which yields contradiction.

(3) G **is edge 3-connected** Suppose not, let $A \subseteq V(G)$ be such that $|\delta(A)| = 2$, say $\delta(A) = \{e, e'\}$. Now $G' = G/e$ is smaller then G, thus it admits a NZ 5-flow f. We extend it to G by letting $f(e) = \pm f(e')$ (the sign is chosen according to the orientation of e, e'). As we saw already in several occasions, this extension yields a flow.

(4) G **is cyclically edge 4-connected** Note that a graph G is called *cyclically edge k-connected*, if $|\delta(A)| \geq k$ whenever A is a set of vertices such that both A and $\bar{A}$ contain a circuit. (Exercise: determine the cyclic edge connectivity of the Petersen graph.)

Suppose G fails the above definition with $k = 3$, that is there is A such that $|\delta(A)| = 3$. Put $G_1 = G/A$, $G_2 = G/\bar{A}$ – both G_1 and G_2 are smaller than G, thus admit a NZ 5-flow. Now it is possible to show [Sekine and Zhang] that

$$F_G(x) = \frac{F_{G_i}(x) \cdot F_{G_2}(x)}{F_{K_2^3}(x)} .$$

(Here $K_2^3 = C_3^*$ is the graph with two vertices and three parallel edges.)

Using this with $x = 5$ (CHECK) gives us that G has a NZ 5-flow, a contradiction.

(5) G **is cyclically edge 6-connected** [Kochol 2004]

(6) G **is has no circuit of length less than 9** [Kochol 2006]

7 Snarks

The above study of the minimal counterexample to 5-flow conjecture suggests to study some nice class of graphs that contains the hypothetical counterexample, so that we either find it, or prove enough properties of the class that would aid us in a proof of the conjecture. A suitable class of such graphs are so-called snarks. Recall, that a graph is called a *snark*, if it is cubic, bridgeless and not 3-edge-colorable. (Equivalently, not having 4-NZF.) Some authors require a higher edge-connectivity (we may insist on the graph to be cyclically 6-edge-connected), but we won't do it here.

Snarks with a 2-cut Start with graphs G and H each with a specified edge. To form the graph $G{=}H$ we cut the specified edges in G and H and glue the "half-edges" to connect G and H – see Fig. **??**. (There are two ways to pair the edges, potentially leading to non-isomorphic graphs $G{=}H$.)

Observe that any edge 3-coloring of $G{=}H$ gives the same color to the two edges of the 2-cut (this is best seen using the equivalence between edge 3-colorings and NZ 4-flows). Consequently, we may use the coloring of $G{=}H$ to get colorings of G and of H. Conversely, for any 3-edge colorings of G and H we may assume that the specified edges in both graphs have the same color, thus we can "glue" the colorings to a coloring of $G{=}H$. To say this observation in another way:

$G{=}H$ is a snark iff G or H is a snark.

Equivalently: when we "add anything to an edge of a snark", we get again a snark (Fig. **??**.)

Snarks with a 3-cut Now we start with cubic graphs G, H each with one specified vertex. We split these specified vertices in three vertices of degree 1, and identify the three pendant of G with those of H. (There are 3! ways to do so.) We use $G{\equiv}H$ to denote the resulting graph.

Given a 3-edge colorings of G and of H, we "rename colors" in H so that the colors of the edges to be identified are the same. This way we obtain a 3-edge coloring of $G{\equiv}H$. On the other hand, in any 3-edge coloring of $G{\equiv}H$ the three edges of the 3-cut have distinct colors, so by contracting whole G (resp. H) to a single vertex, we get a proper edge coloring of H (resp. G). So we obtain:

$G{\equiv}H$ is a snark iff G or H is a snark.

Equivalently: when we "add anything to a vertex of a snark", we get again a snark.

Exercise: We define two useful operations on cubic graphs. A Δ-Y transformation is a contraction of a triangle to a single vertex, a Y-Δ transformation is the inverse operation. (Observe that these operation preserve the 3-regularity.) For a cubic graph G, prove that G is a snark iff G' obtained by a series of Y-Δ and Δ-Y transformations from G is a snark.

Notes: : The simplicity of the above two constructions, in particular the fact that only one of the smaller graphs needs to be a snark, together with possibility to reduce the "big conjectures" to cyclically 4-edge-connected graphs, explain why many authors choose to demand that the snarks are free of 2-cuts and non-trivial 3-cuts.

Snarks with a 4-cut – Isaacs' dot product Let G, H be graphs, ab, cd edges of G, e an edge of H, let x, y be the other two neighbours of one end of e, u, v the other two neighbours of the other end. To form the *Isaacs' dot product* $G \cdot H$ of G and H we delete edges ab and cd from G, e with its end-vertices from H, and add edges ax, by, cu, dv. (See Fig. **??**.)

Theorem 29 (Isaacs, 1975)**.** *If G and H are snarks then so is $G \cdot H$. If both G and H are cyclically 4-edge-connected, then so is $G \cdot H$.*

Proof. Suppose we have an edge 3-coloring f of $G \cdot H$. We distinguish two cases.

(1) $f(ax) = f(by)$ It follows that also $f(cu) = f(dv)$. We may restrict f to the edges of G, putting $f(ab) = f(ax)$, and $f(cd) = f(cu)$. Easily, this is a proper 3-edge-coloring, a contradiction.

(2) $f(ax) \neq f(by)$ Again, we use the fact that f (treated as a 4-NZF) sums to zero over edges ax, by, cu, dv, to conclude that $f(cu), f(dv)$ are the same two distinct values as $f(ax), f(by)$. This implies that the restriction of f to the edges of H can be extended to the edge e, yielding a proper edge 3-coloring, a contradiction.

It remains to establish the cyclic 4-edge connectivity. Let $A \subseteq V(G \cdot H)$ be such that $|\delta(A)| = 4$ and both A and $\bar{A}$ induce a cycle.

□

Example: Blanuša, etc.

Flower snarks Let n be odd. To describe a graph J_n, we start with three copies of C_n, we denote its vertices by i_1, i_2, i_3 for $i = 1, \dots, n$. Replace edges $n_2 1_2$ and $n_3 1_3$ by $n_2 1_3$ and $n_3 1_2$. Finally, for each i we add a new vertex i and join it by an edge to i_1, i_2, i_3. On Figure **??** we can see J_5 (this particular graph is sometimes called the flower snark). and J_3 — is just a Y-Δ transformation of Pt (equivalently, it is Pt$\equiv K_4$).

Theorem 30 (Isaacs, 1975)**.** *If n is odd then J_n is a snark. If $n \geq 7$ then J_n is cyclically 6-edge-connected.*

Proof. Suppose J_n can be edge-colored using three colors. Let B_i denote the subgraph induced by vertices i, i_1, i_2, i_3 and the incident edges (see Fig. **??**). We divide the edges of this subgraph into three triples, Left, Right, and Top. (Of course the Right edges of B_i are the Left edges of B_{i+1}.) Clearly not all edges of of L can be of the same color, as then it is not possible to color T. Thus there are two possibilities.

(1) Edges of L use one color twice. Say, they use colors 1, 1, and 2 in some order. It is easy to check that then edges of R use colors 2, 3, and 3, in some order. In the next block we will use 1, 1, 2 on the right, and so on. As n is odd, we get a contradiction.

(2) Edges of L use all three colors. Again, it is simple to explore the two possibilities how to extend the coloring on R: both are obtained from the coloring of L by a cyclic shift (i.e., a permutation formed by one 3-cycle). In between the blocks B_n and B_1 we introduced a transposition by the construction of the graph. Thus if there is an edge 3-coloring, then we can write an identity as a composition of 3-cycles and one transposition, which is a contradiction.

TODO: cyclic connectivity?

□