A graph G is *f-treewidth-fragile* if for every integer k, there exists a partition of $V(G)$ to parts $X_1, \ldots, X_k$ such that $\mathrm{tw}(G - X_i) \leq f(k)$ for $i = 1, \ldots, k$. A graph class $\mathcal{G}$ is *f-treewidth-fragile* if every graph in $G \in \mathcal{G}$ is f-treewidth-fragile. A class $\mathcal{G}$ is *treewidth-fragile* if it is f-treewidth-fragile for some function f, and it is *effectively treewidth-fragile* if there exists a polynomial-time algorithm taking $G \in \mathcal{G}$ and $k \geq 1$ as an input and outputing the corresponding partition $X_1, \ldots, X_k$. Our goal is to show that all proper minor-closed classes are (effectively) treewidth-fragile, and to give some applications.

1 Applications of treewidth-fragility

A property π of graphs is *a-hereditary* if for every graph G having the property π and for every $X \subseteq V(G)$, there exists $Y \subseteq V(G)$ such that $X \subseteq Y$ and $|Y| \leq a|X|$ and $G - Y$ has the property π. For example,

- the properties "G has no edges" and "G is 3-colorable" are 1-hereditary, and
- the property "G can be covered by vertex-disjoint triangles" is 3-hereditary.

Let $\alpha_\pi(G)$ denote the size of the largest set $X \subseteq V(G)$ such that $G[X]$ has the property π. We say that π is *tractable in graphs of bounded treewidth* if for every b, there exists a polynomial-time algorithm determining α_π for graphs of treewidth at most b.

Lemma 1. *Suppose a class $\mathcal{G}$ of graphs is effectively treewidth-fragile and a property π is a-hereditary for some $a \geq 1$ and tractable in graphs of bounded treewidth. Then for every $p \geq 1$, there exists a polynomial-time algorithm that for a graph $G \in \mathcal{G}$ returns $Z \subseteq V(G)$ such that $G[Z]$ has the property π and $|Z| \geq (1 - 1/p)\alpha_\pi(G)$.*

Proof. Without loss of generality, we can assume a and p are integers (by rounding them up if necessary). Let f be the function such that every graph from $\mathcal{G}$ is f-treewidth-fragile. Let $k = ap$. In polynomial time, we can find a partition $X_1, \ldots, X_k$ of $V(G)$ such that $\mathrm{tw}(G_i) \leq f(k)$ for $i = 1, \ldots, k$. For $i = 1, \ldots, k$, use the algorithm for bounded treewidth to find $Z_i \subseteq V(G - X_i)$ of size $\alpha_\pi(G - X_i)$ such that $G[Z_i]$ has the property π, and return the largest set Z among $Z_1, \ldots, Z_k$.

Consider a set $T \subseteq V(G)$ such that $G[T]$ has property π and $|T| = \alpha_\pi(G)$. For $i = 1, \ldots, k$, let $X'_i = X_i \cap T$; there exists i such that $|X'_i| \leq |T|/k$. Since π is a-hereditary and $G[T]$ has the property π, there exists a set $X''_i \subseteq T$

such that $X'_i \subseteq X''_i$, $|X''_i| \le a|X'_i| \le |T|/p$, and $G[T \setminus X''_i]$ has the property π. Since $T \setminus X''_i \subseteq V(G) \setminus X_i$, we have

$$|Z| \ge |Z_i| = \alpha_\pi(G - X_i) \ge |T \setminus X''_i| \ge (1 - 1/p)|T| = (1 - 1/p)\alpha_\pi(G),$$

as required. □

Lemma 2. *Suppose a class $\mathcal{G}$ of graphs is effectively treewidth-fragile. Then the chromatic number can be approximated for graphs in $\mathcal{G}$ up to a factor of 2.*

Proof. Let f be the function such that every graph from $\mathcal{G}$ is f-treewidth-fragile. For $G \in \mathcal{G}$, let X_1, X_2 be a partition of $V(G)$ such that $\mathrm{tw}(G - X_1), \mathrm{tw}(G - X_2) \le f(2)$. Color the graphs $G - X_1$ and $G - X_2$ optimally by disjoint sets of colors, obtaining a coloring of G by

$$\chi(G - X_1) + \chi(G - X_2) \le \chi(G) + \chi(G) = 2\chi(G)$$

colors. □

2 Graphs on surfaces

Lemma 3. *Suppose G is a graph drawn on a surface of Euler genus g. If G has radius r, then $tw(G) \le (2g + 3)r$.*

Proof. Without loss of generality, G is a triangulation. Applying BFS to G, we obtain a rooted spanning tree T of G of depth r; let q be the root of T and for each vertex $v \in V(G)$, let $t(v)$ denote the set of at most r vertices on the path from v to q in T, including v but excluding q. Let $G^\star$ be the dual of G, and let S be the spanning subgraph of $G^\star$ whose edges correspond to those in $E(G) \setminus E(T)$. Each vertex f of $G^\star$ corresponds to a face of G, bounded by a cycle xyz; let us define $t(f) = t(x) \cup t(y) \cup t(z)$ and note that $|t(f)| \le 3r$.

Note the graph S is connected. Indeed, we can "walk around" the tree T in G, passing along edges of S and visiting all faces of G (vertices of S). Let S_0 be a spanning tree of S and let $X = E(S) \setminus E(S_0)$. We have

$$\begin{aligned} |X| &= |E(S)| - |E(S_0)| = (|E(G)| - |E(T)|) - |E(S_0)| \\ &= |E(G)| - (|V(G)| - 1) - (|V(G^\star)| - 1) \\ &= (|V(G)| + |V(G^\star)| + g - 2) - (|V(G)| - 1) - (|V(G^\star)| - 1) = g. \end{aligned}$$

Let X' be the set of vertices of G incident with the edges corresponding to X, and let $Z = \bigcup_{v \in X'} t(v)$; we have $|Z| \le 2gr$. For $f \in V(G^\star)$, let us define

$\beta(f) = t(f) \cup Z \cup \{q\}$; we have $\beta(f) \le (2g+3)r + 1$. Hence, it suffices to argue that (S_0, β) is a tree decomposition of G.

For any edge $uv \in E(G)$, we have $\{u, v\} \subseteq \{q\} \cup t(u) \cup t(v) \subseteq \beta(f)$ for a face $f \in V(S_0)$ incident with this edge. Consider any vertex $v \in V(G)$. If $v \in \{q\} \cup Z$, then v appears in all bags of (S_0, β). Otherwise, let T_v be the subtree of T rooted in v, and note that $v \in \beta(f)$ exactly for the faces f incident with vertices of T_v. Any two such faces are connected by a walk in S obtained by "walking around" T_v; the edges of this walk must belong to S_0, since $v \notin Z$ implies no edge of S corresponding to an edge of G incident with a vertex of T_v belongs to X. Therefore, $\{f : v \in \beta(f)\}$ induces a connected subtree of S_0. □

3 Outgrowths

Recall:

Definition 4. *A graph H is a* vortex of depth d *and* boundary sequence $v_1, \ldots, v_k$ *if H has a path decomposition (T, β) of width at most d such that*

- *$T = v_1 v_2 \ldots v_k$, and*
- *$v_i \in \beta(v_i)$ for $i = 1, \ldots, k$*

Definition 5. *For G_0 drawn in a surface, a graph G is an* outgrowth of G_0 by m vortices of depth d *if*

- *$G = G_0 \cup H_1 \cup H_m$, where $H_i \cap H_j = \emptyset$ for distinct i and j,*
- *for all i, H_i is a vortex of depth d intersecting G only in its boundary sequence,*
- *for some disjoint faces $f_1, \ldots, f_k$ of G_0, the boundary sequence of H_i appears in order on the boundary of f_i.*

Let us now generalize Lemma 3.

Lemma 6. *Suppose G is an outgrowth of graph G_0 drawn on a surface of Euler genus g by (any number of) vortices of depth d. If G has radius r, then tw$(G) < (2(2g+3)r+1)(d+1)$.*

Proof. Let $f_1, \ldots, f_k$ be the faces of G_0 to which the vortices $G_1, \ldots, G_k$ attach. For $i = 1, \ldots, k$, let (T_i, β_i) be the corresponding decomposition of G_i; we can assume T_i is a path in G_0. Let G_0' be obtained from G_0 by, for $i = 1, \ldots, k$, adding a vertex adjacent to all vertices incident with f_i; note that G_0' has radius at most $2r$. Let (T, β_0) be the tree decomposition of G_0 obtained by Lemma 3; we have $|\beta(x)| \le 2(2g+3)r+1$ for $x \in V(T)$. For $v \in V(G_0)$, if there exists (necessarily unique) index i such that $v \in V(T_i)$, let $\alpha(v) = \beta_i(v)$, otherwise let $\alpha(v) = \{v\}$. For $x \in V(T)$, let $\beta(x) = \bigcup_{v \in \beta_0(x)} \alpha(v)$. Then (T, β) is a tree decomposition of G of width less than $(2(2g+3)r+1)(d+1)$.

Indeed, consider any $v \in V(G)$. If there exists i such that $v \in V(G_i)$, then there exists a connected subpath $T_v \subseteq G_0$ of T_i such that $v \in \beta_i(x)$ exactly for $x \in V(T_v)$, and let T_v' be the connected subtree of T induced by the vertices x such that $\beta_0(x) \cap V(T_v) \neq \emptyset$; otherwise, let $T_v' = \varnothing$. If $v \in V(G_0)$, then let T_v'' be the connected subtree of T induced by the vertices x such that $v \in \beta_0(x)$; otherwise, let $T_v'' = \varnothing$. Note that $\{x \in V(T) : v \in \beta(x)\} = V(T_v' \cup T_v'')\}$, and that if $T_v' \neq \varnothing \neq T_v''$, then $v \in V(T_i)$, and thus $T_v' \cap T_v'' \neq \emptyset$, implying that $T_v' \cup T_v''$ is connected. □

Let $\mathcal{G}_{g,d}$ be the class of outgrowths of graphs drawn on a surface of Euler genus g by (any number of) vortices of depth d. For a vortex with decomposition (T, β), a vertex x is *boundary-universal* if it is adjacent to all vertices of T. Let $\mathcal{G}_{g,d}'$ be the class of outgrowths of graphs drawn on a surface of Euler genus g by (any number of) vortices of depth d, each of them containing a boundary-universal vertex.

Corollary 7. *For any g, d, b, r, consider a graph $G \in \mathcal{G}_{g,d}'$. If Z is the set of vertices of G at distance at least b and at most $b+r$ from some vertex v_0 in the embedded part of G, then tw$(G[Z]) < (2(2g+3)(r+5)+1)(d+1)$*

Proof. Without loss of generality, we can assume G is connected. Let G_0 be the embedded part of G. For each vortex G_i of G, let (T_i, β_i) be the corresponding decomposition. Let H be obtained from G as follows. Delete all vertices at distance greater than $b+r$ from v_0 that are not in the boundary of any vortex, except for the boundary-universal vertices at distance exactly $b+r+1$ from v_0. For each vortex G_i,

(a) if all vertices of T_i are at distance greater than $b+r$ from v_0, then delete $V(T_i)$, and

(b) if all vertices of T_i are at distance less than b from v_0, then contract G_i to a single vertex and do not consider it to be a vortex any more.

Finally, contract all edges joining vertices u and v at distance less than b from v_0 such that at least one of u and v is not contained in a boundary of a vortex.

Let H' be the subgraph of G induced by vertices at distance at least b and at most $b+r$ from v_0 that are contained in vortices G_i such that all vertices of T_i are at distance less than b from v_0 (i.e., the vortices eliminated in (b) above). Note that H' is a union of components of $G[Z]$, treewidth of H' is less than d, and $G[Z \setminus V(H')]$ is a subgraph of H. Hence, it suffices to argue that $\mathrm{tw}(H) < (2(2g+3)(r+5)+1)(d+1)$. Note also that $H \in \mathcal{G}_{g,d}$, and thus by Lemma 6, it suffices to argue H has radius at most $r+5$

Indeed, consider any vertex $v' \in V(H)$, and let v be one of vertices of G which have been contracted to v'. Let P be a shortest path from v_0 to v in G; the construction of H and the fact that vortices contain boundary-universal vertices implies that P has length at most $b+r+2$. Consider any edge xy of P, where both x and y are at distance less than $b-2$ from v_0. If one of these vertices is a boundary vertex of a vortex, then since the vortex contains a boundary-universal vertex, all the vertices of the boundary are at distance less than b from v_0, and thus the vortex was contracted in (b) to a single vertex. Otherwise, the edge xy was contracted in the last part of the construction of H. Therefore, P is contracted to a path of length at most $r+5$. □

Corollary 8. *For every g and d, then class $\mathcal{G}_{g,d}$ is treewidth-fragile.*

Proof. Consider a graph $G \in \mathcal{G}_{g,d}$; without loss of generality, we can assume G is connected. For each vortex G_i of G, let (T_i, β_i) be the corresponding decomposition, and let G' be obtained by, for each i, adding a vertex v_i adjacent to all vertices of T_i to the graph and putting v_i to all bags of β_i; clearly, $G \in \mathcal{G}'_{g,d+1}$. Let v_0 be an arbitrary vertex of the embedded part of G'.

Consider any integer $k \geq 1$. For $i = 1, \ldots, k$, let X'_i consist of vertices whose distance from v_0 in G' modulo k is $i-1$, and let $X_i = X'_i \cap V(G)$. It suffices to argue that the treewidth of $G' - X'_i \supseteq G - X_i$ is bounded. This follows from Corollary 7, since $G' - X'_i$ is a disjoint union of subgraphs induced by vertices at distances at least $tk+i$ and at most $tk+i+k-1$ for $t \in \mathbb{Z}$. □

4 Apices and clique-sums

Recall:

Definition 9. *G is obtained from H by* adding a apices *if $H = G - A$ for some set $A \subseteq V(G)$ of size a.*

For a class $\mathcal{G}$, let $\mathcal{G}^{(a)}$ denote the class of graphs obtained from those in $\mathcal{G}$ by adding at most a apices. For a function f, let $f^{(a)}(k) = f(k) + a$.

Observation 10. *If $\mathcal{G}$ is f-treewidth-fragile, then $\mathcal{G}^{(a)}$ is $f^{(a)}$-treewidth-fragile.*

Proof. Consider a graph $G \in \mathcal{G}^{(a)}$, and let A be a set of size at most a such that $G - A \in \mathcal{G}$. Let $X'_1, \ldots, X'_k$ be a partition of $V(G - A)$ such that $\text{tw}(G - A - X'_i) \leq f(k)$ for each i. Let $X_1 = X'_1, \ldots, X'_{k-1} = X_{k-1}$, $X_k = X'_k \cup A$. Then $\text{tw}(G - X_i) \leq f(k) + a$ for each i. □

Observation 11. *If $\mathcal{G}$ is f-treewidth-fragile, then $\omega(G) \leq 2f(2)+2$ for every $G \in \mathcal{G}$.*

Proof. Let X_1, X_2 be a partition of $V(G)$ such that $\text{tw}(G-X_1), \text{tw}(G-X_2) \leq f(2)$. Then

$$\omega(G) \leq \omega(G-X_1)+\omega(G-X_2) \leq (\text{tw}(G-X_1)+1)+(\text{tw}(G-X_2)+1) \leq 2f(2)+2.$$

□

Lemma 12. *Let $\mathcal{G}$ be a class of graphs and let $\mathcal{H}$ be the class of graphs obtained from those in $\mathcal{G}$ by clique-sums. If $\mathcal{G}$ is f-treewidth-fragile, then $\mathcal{H}$ is $f^{(2f(2)+2)}$-treewidth-fragile.*

Proof. Note that for every $H \in \mathcal{H}$, we have $\omega(H) \leq 2f(2) + 2$. Consider any $k \geq 1$. We will inductively show a stronger claim: For every $H \in \mathcal{H}$ and a partition $K_1, \ldots, K_k$ of a clique K in H, there exists a partition $X_1, \ldots, X_k$ of H such that $\text{tw}(H - X_i) \leq f(k) + 2f(2) + 2$ and $K \cap X_i = K_i$ for each i. This is clear for graphs $G \in \mathcal{G}$: Take the partition obtained by f-treewidth-fragility of G and move all vertices of K to the appropriate part, increasing the treewidth of $G - X_i$ by at most $|K|$.

Suppose we now perform a clique-sum of $H_1, H_2 \in \mathcal{H}$ on a clique Q, to obtain a graph H, and let K be a clique in H and $K_1, \ldots, K_k$ its partition. We can by symmetry assume $K \subseteq V(H_1)$. Let $X'_1, \ldots, X'_k$ be the inductively obtained partition of $V(H_1)$ such that $\text{tw}(H_1 - X'_i) \leq f(k) + 2f(2) + 2$ and $K \cap X'_i = K_i$ for each i. Let $Q_i = Q \cap X'_i$ for each i, and let $X''_1, \ldots, X''_k$ be the inductively obtained partition of $V(H_2)$ such that $\text{tw}(H_2 - X''_i) \leq f(k) + 2f(2) + 2$ and $Q \cap X''_i = Q_i$ for each i. Letting $X_i = X'_i \cup X''_i$, we obtain a partition of $V(H)$ such that $K \cap X_i = K_i$ for each i. Moreover, $H - X_i$ is a clique-sum of $H_1 - X'_i$ and $H_2 - X''_i$, implying $\text{tw}(H - X_i) \leq f(k) + 2f(2) + 2$. □

5 Proper minor-closed classes

Recall:

Definition 13. *A graph G is a-*near-embeddable *in a surface Σ if for some graph G_0 drawn in Σ, G is obtained from an outgrowth of G_0 by at most a vortices of depth a by adding at most a apices.*

Theorem 14 (The Structure Theorem)**.** *For every proper minor-closed class $\mathcal{G}$, there exists a and g such that graphs in $\mathcal{G}$ are clique-sums of graphs that are a-near-embeddable in surfaces of genus at most g.*

Combining the structure theorem with Lemma 12, Observation 10, and Corollary 8, we obtain the following claim.

Corollary 15. *Every proper minor-closed class is treewidth-fragile.*