Definition

A graph G is f-treewidth-fragile if for every integer $k \geq 1$, there exists a partition $X_1, \ldots, X_k$ of $V(G)$ such that

$$\mathrm{tw}(G - X_i) \leq f(k)$$

for $i = 1, \ldots, k$.

Application: Subgraph testing

Lemma

$H \subseteq G$ for a graph G of treewidth at most t can be decided in time $O(t^{|H|}|G|)$.

Observation

For $k = |H| + 1$, if $H \subseteq G$, then there exists i such that $V(H) \cap X_i = \emptyset$.

Corollary

Deciding $H \subseteq G$ in time

$$O(kf(k)^{|H|}|G|$$

by testing $H \subseteq G - X_1, \ldots, H \subseteq G - X_k$.

Application: Chromatic number approximation

Lemma

Optimal coloring of a graph G of treewidth t can be obtained in time $O((t+1)^{t+1}|G|)$.

Corollary

Coloring by $\leq 2\chi(G)$ colors in time $O((f(2)+1)^{f(2)+1}|G|)$: use disjoint sets of colors on $G - X_1$ and $G - X_2$.

Application: Triangle matching

$\mu_3(G) =$ maximum number of vertex-disjoint triangles in G.

Lemma

Triangle matching of size $\mu_3(G)$ can be found in time $O(4^t(t+1)!|G|)$ for a graph G of treewidth t.

Observation

For some i, X_i intersects at most $3\mu_3(G)/k$ of the optimal solution triangles $\Rightarrow \mu_3(G - X_i) \geq (1 - 3/k)\mu_3(G)$.

Corollary

Triangle matching of size $(1 - 3/k)\mu_3(G)$ can be found in time $O(f(k)4^{f(k)}(f(k)+1)!|G|)$: Return largest of results for $G - X_1$, $\ldots$, $G - X_k$.

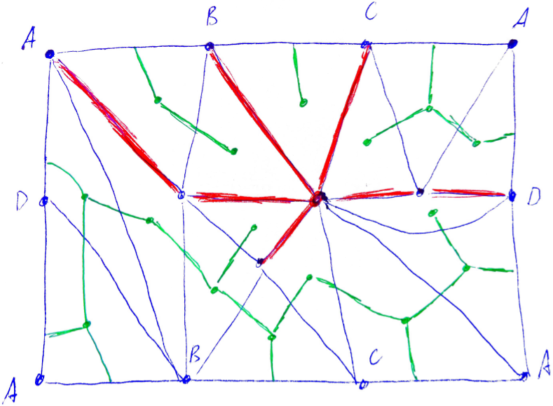
How to prove things for proper minor-closed classes:

- solve bounded genus and bounded treewidth case
- extend to graphs with vortices
- incorporate apex vertices
- deal with clique-sums/tree decomposition

Lemma

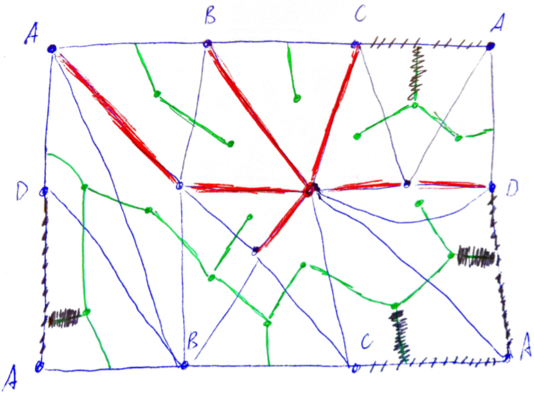
G has genus g, radius r $\Rightarrow tw(G) \leq (2g+3)r$.

- WLOG G is a triangulation: dual $G^\star$ is 3-regular.
- T BFS spanning tree of G
- S spanning subgraph of $G^\star$ with edges $E(G) \setminus E(T)$.



- S is connected; S_0: a spanning tree of S, $X^\star = E(S) \setminus E(S_0)$

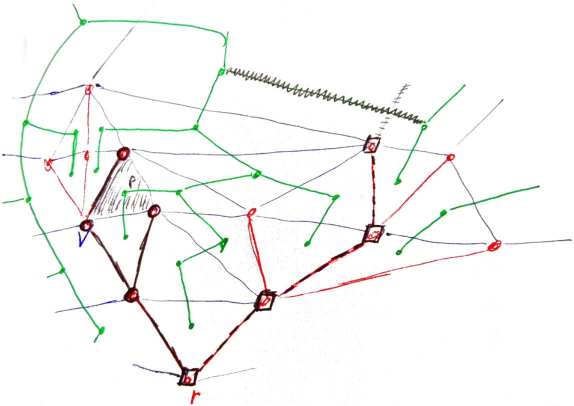
$$
\begin{aligned}
|X^\star| &= |E(S)| - |E(S_0)| = (|E(G)| - |E(T)|) - |E(S_0)| \\
&= |E(G)| - (|V(G)| - 1) - (|V(G^\star)| - 1) \\
&= (|V(G)| + |V(G^\star)| + g - 2) - (|V(G)| + |V(G^\star)| - 2) = g.
\end{aligned}
$$



- $t(v)$ = vertices on path from v to root in T.
- X: edges of G corresponding to $X^\star$.
- For $f \in V(G^\star)$,

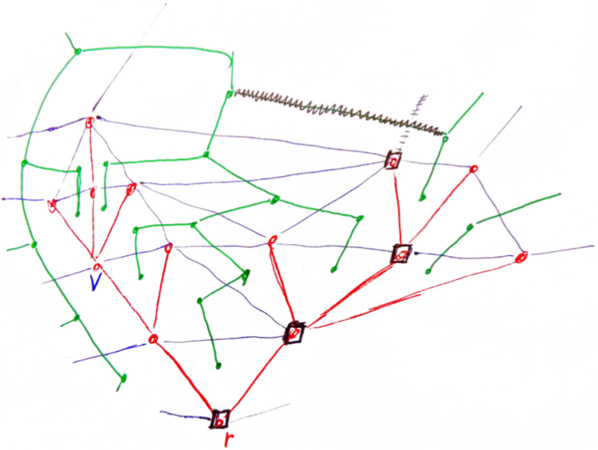
$$\beta(f) = \bigcup_{v \text{ incident with } f \text{ or } X} t(v)$$

- $|\beta(f)| \leq (2g+3)r + 1$



(S_0, β) is a tree decomposition:

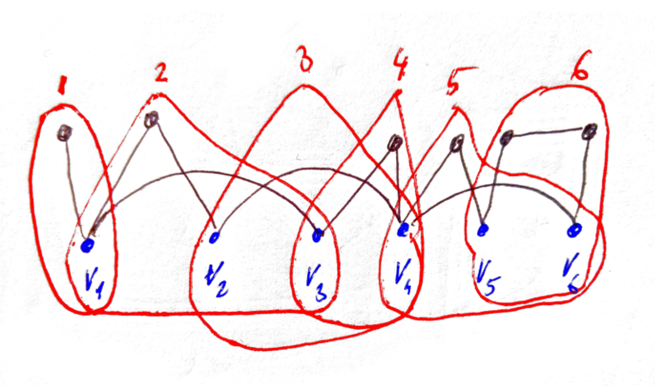
- f incident with uv: $\{u, v\} \subseteq t(u) \cup t(v) \subseteq \beta(f)$.
- T_v subtree of T rooted in v:
  - T_v incident with edge of $X \Rightarrow v$ in all bags.
  - Otherwise: Walking around T_v shows $S_0[\{x : v \in \beta(x)\}]$ is connected.



Definition

A graph H is a vortex of depth d and boundary sequence $v_1, \ldots, v_k$ if H has a path decomposition (T, β) of width at most d such that

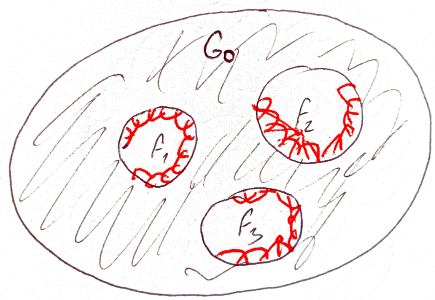
- $T = v_1 v_2 \ldots v_k$, and
- $v_i \in \beta(v_i)$ for $i = 1, \ldots, k$



Definition

For G_0 drawn in a surface, a graph G is an **outgrowth of G_0 by m vortices of depth d** if

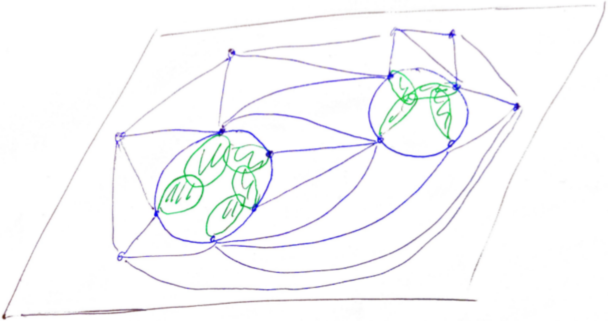
- $G = G_0 \cup H_1 \cup H_m$, where $H_i \cap H_j = \emptyset$ for distinct i and j,
- for all i, H_i is a vortex of depth d intersecting G only in its boundary sequence,
- for some disjoint faces $f_1, \ldots, f_k$ of G_0, the boundary sequence of H_i appears in order on the boundary of f_i.



Lemma

G outgrowth of graph G_0 of Euler genus g by vortices of depth d, radius $r \Rightarrow tw(G) < (2(2g+3)r+1)(d+1)$.

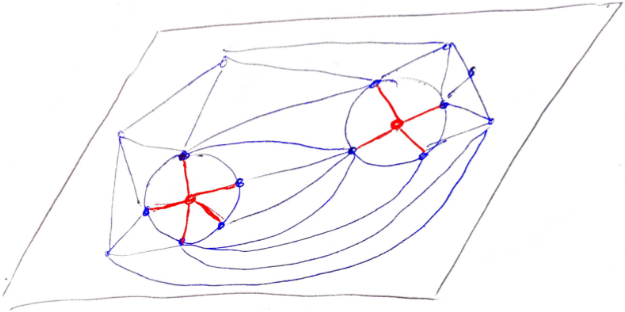
- (T_i, β_i) decomposition of a vortex: WLOG T_i a path in G_0.
- G_0': shrink interiors of vortices to single vertices; radius$(G_0') \leq 2r$
- (T, β_0): Tree decomposition of G_0' of width $2(2g+3)r$.
- For $v \in V(T_i)$: Replace v by $\beta_i(v)$ in bags of (T, β_0).



Lemma

G outgrowth of graph G_0 of Euler genus g by vortices of depth d, radius r $\Rightarrow$ $tw(G) < (2(2g+3)r+1)(d+1)$.

- (T_i, β_i) decomposition of a vortex: WLOG T_i a path in G_0.
- G_0': shrink interiors of vortices to single vertices; radius$(G_0') \leq 2r$
- (T, β_0): Tree decomposition of G_0' of width $2(2g+3)r$.
- For $v \in V(T_i)$: Replace v by $\beta_i(v)$ in bags of (T, β_0).

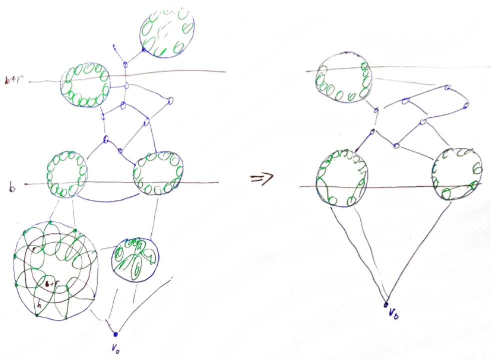


Vortex G_i is <u>local</u> if $d_{G_i}(x, y) \leq 2$ for each $x, y \in V(T_i)$.

Corollary (Layer Corollary)

G outgrowth of graph G_0 of Euler genus g by local vortices of depth d, Z vertices at distance $b, \ldots, b+r$ from $v_0 \in V(G_0) \Rightarrow tw(G) < (2(2g+3)(r+5)+1)(d+1)$.

- Delete vortices at distance $> b+r$, non-boundary vertices at distance $> b+r+1$
- Shrink vortices at distance $< b-2$.
- Contract edges between vertices at distance $< b-2 \Rightarrow$ radius $\leq r+5$.

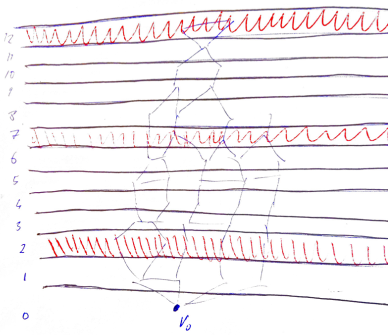


$\mathcal{G}_{g,d}$: outgrowths of graphs of Euler genus g by vortices of depth d.

Corollary

$\mathcal{G}_{g,d}$ is f-treewidth-fragile for $f(k) = (2(2g+3)(k+5)+1)(d+2)$.

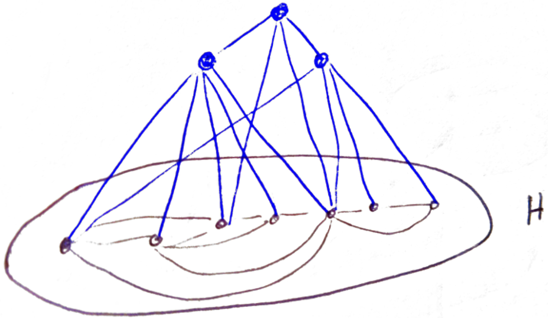
- Add a universal vertex to each vortex to make it local.
- Let $X_i = \{v : d(v_0, v) \bmod k = i\}$ for $i = 0, \ldots, k-1$.
- Layer Corollary applies to each component of $G - X_i$.



Definition

G is obtained from H by adding a apices if $H = G - A$ for some set $A \subseteq V(G)$ of size a.

$\mathcal{G}^{(a)}$ = graphs obtained by adding at most a apices to graphs from $\mathcal{G}$.



Lemma

$\mathcal{G}$ is f-treewidth-fragile $\Rightarrow \mathcal{G}^{(a)}$ is h-treewidth-fragile for $h(k) = f(k) + a$.

Proof.

Add the apex vertices to X_1. □

Lemma

$\mathcal{G}$ is f-treewidth-fragile $\Rightarrow \omega(G) \leq 2f(2) + 2$ for $G \in \mathcal{G}$.

Proof.

$$\omega(G) \leq \omega(G - X_1) + \omega(G - X_2) \leq 2f(2) + 2.$$

□

For a partition $K_1, \dots, K_k$ of $K \subseteq V(G)$, a partition $X_1, \dots, X_k$ of $V(G)$ extends it if $K_i = K \cap X_i$ for $i = 1, \dots, k$.

Definition

$\mathcal{G}$ is strongly f-treewidth-fragile if for every $G \in \mathcal{G}$, every $k \geq 1$, and every clique K in G, every partition of K extends to a partition $X_1, \dots, X_k$ of $V(G)$ such that $\mathrm{tw}(G - X_i) \leq f(k)$ for $i = 1, \dots, k$.

Lemma

$\mathcal{G}$ is f-treewidth-fragile $\Rightarrow$ $\mathcal{G}$ is strongly h-treewidth-fragile for $h(k) = f(k) + 2f(2) + 2$.

Proof.

Re-distribute the vertices of K, increasing treewidth by $\leq |K| \leq 2f(2) + 2$. □

Lemma

$\mathcal{G}$ is strongly f-treewidth-fragile $\Rightarrow$ clique-sums of graphs from $\mathcal{G}$ are strongly f-treewidth-fragile.

Proof.

- G clique-sum of G_1 and G_2 on a clique Q, $K \subseteq V(G)$.
- WLOG $K \subseteq G_1$.
- Extend the partition of K to a partition $X'_1, \ldots, X'_k$ of G_1.
- Extend the partition $Q \cap X'_1, \ldots, Q \cap X'_k$ to a partition $X''_1, \ldots, X''_k$ of G_2.
- Let $X_i = X'_i \cup X''_i$; $G - X_i$ is a clique-sum of $G_1 - X'_i$ and $G_2 - X''_i$:

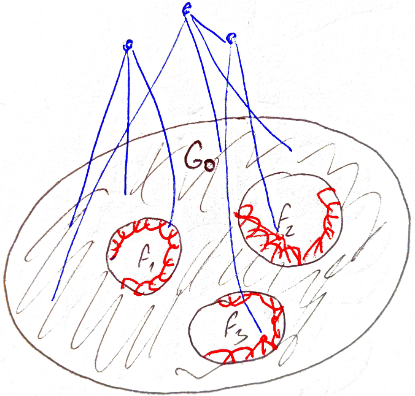
$$\operatorname{tw}(G - X_i) = \max(\operatorname{tw}(G_1 - X'_i), \operatorname{tw}(G_2 - X''_i)) \leq f(k).$$

□

Near-embeddability

Definition

A graph G is a-near-embeddable in a surface Σ if for some graph G_0 drawn in Σ, G is obtained from an outgrowth of G_0 by at most a vortices of depth a by adding at most a apices.



Theorem (The Structure Theorem)

For every proper minor-closed class $\mathcal{G}$, there exist g and a such that every graph in $\mathcal{G}$ is obtained by clique-sums from graphs a-near-embeddable in surfaces of genus at most g.

Corollary

For every proper minor-closed class $\mathcal{G}$, there exists a linear function f such that $\mathcal{G}$ is f-treewidth-fragile.