Our goal for this lecture is to present a (very brief) outline of the proof of the structure theorem. First, we need to formulate its local version with respect to a given tangle.

Let G be a graph and let Ω be a cyclic sequence of vertices of G. Then (G, Ω) is a *society*. We view a graph itself as a society with empty sequence. A *cross* in a society consists of two disjoint paths P_1 and P_2 in G such that the labels of the ends x_1 and y_1 of P_1 and x_2 and y_2 of P_2 can be chosen so that they appear in Ω in order x_1, x_2, y_1, and y_2. A society is a *cell* if $|\Omega| \le 3$. A *transaction* of order p in the society (G, Ω) is a set of p pairwise vertex-disjoint paths with ends in Ω. A society is a *p-vortex* if it contains no transaction of order greater than p; from the homework assignment, we have the following description of p-vortices (recall the *adhesion* of a tree decomposition (T, β) is the maximum of $|\beta(x) \cap \beta(y)|$ over distinct $x, y \in V(T)$).

Lemma 1. *If (G, Ω) is a p-vortex and $\Omega = (v_1, \ldots, v_m)$, then G has a path decomposition (P, β) over the path $P = v_1 v_2 \ldots v_m$ of adhesion at most p such that $v_i \in \beta(v_i)$ for $i = 1, \ldots, m$.*

A society (G_1, Ω_1) is a *subsociety* of (G, Ω) if G_1 is a subgraph of G, every edge of G_1 incident with $V(G_1) \setminus \Omega_1$ belongs to G_1, and $G_1 \cap \Omega \subseteq \Omega_1$. Two subsocieties (G_1, Ω_1) and (G_2, Ω_2) are *disjoint* if $G_1 \cap G_2 = \Omega_1 \cap \Omega_2$. A *segregation* of (G, Ω) is a set $\{(G_i, \Omega_i) : i = 1, \ldots, n\}$ of its disjoint subsocieties such that $G = G_1 \cup \ldots \cup G_n$. The segregation is *of type* (k, p) if all but at most k elements are cells and the remaining at most k elements are p-vortices.

If $\Omega = \emptyset$, an *arrangement* of the segregation in a surface Σ is a function α satisfying the following conditions: $\alpha(G_i, \Omega_i)$ is a disk $\Delta_i \subseteq \Sigma$ and for each $v \in \Omega_i$, $\alpha(v)$ is a distinct point in Σ contained in the boundary of Δ_i, such that

- for each i, the order of the points $\alpha(v)$ for $v \in \Omega_i$ in the boundary of Δ_i matches the order of the vertices v in Ω_i, and
- for distinct i and j, the disks Δ_i and Δ_j intersect exactly in the points $\alpha(v)$ for $v \in \Omega_1 \cap \Omega_2$.

If Ω is not emptyset, we additionally require Σ has exactly one hole and

- for each $v \in \Omega$, the point $\alpha(v)$ is contained in the boundary of Σ and their order in the boundary matches the order of the vertices v in Ω.

A society is *rural* if it has a segregation into cells with an arrangement in a disk. In the homework assignment, we have seen the following result.

Lemma 2. *A society (G, Ω) is rural if and only if it does not contain a cross.*

For a tangle $\mathcal{T}$ in G of order θ, we say that a segregation $\{(G_i, \Omega_i) : i = 1, \ldots, n\}$ of G is *$\mathcal{T}$-central* if there is no $(A, B) \in \mathcal{T}$ and $i \in \{1, \ldots, n\}$ such that $B \subseteq G_i$. For $Z \subseteq V(G)$ with $|Z| < \theta$, recall that we can naturally define a tangle $\mathcal{T} - Z$ in $G - Z$ of order $\theta - |Z|$ as the set of all separations $\{(A - Z, B - Z) : (A, B) \in \mathcal{T}, Z \subseteq V(A \cap B)\}$.

Theorem 3 (The Structure Theorem, local version)**.** *For every graph F, there exist integers $\alpha < \theta$, k, and p such that the following holds. For every graph G and a tangle $\mathcal{T}$ in G of order at least θ, if $F \not\preceq G$, then there exists $A \subseteq V(G)$ of size at most α, a surface Σ in which F cannot be drawn, and a $(\mathcal{T} - A)$-central segregation of $G - A$ of type (k, p) with an arrangement in Σ.*

1 Global structure theorem from the local one

A graph G is *(b, k, ρ)-near-embedded* in a surface Σ if for some subset $B \subseteq V(G)$, the graph $G - B$ has a drawing in Σ with at most k vortices of width at most ρ. The final global form of the structure theorem we aim for is as follows.

Theorem 4 (The Structure Theorem, local version)**.** *For every graph F, there exist integers b, k, and ρ such that the following holds. For every graph G, if $F \not\preceq G$, then G has a tree decompositions whose torsos can be (b, k, ρ)-near-embedded in surfaces in which F cannot be drawn. Equivalently, G can be obtained from graphs (b, k, ρ)-near-embedded in surfaces in which F cannot be drawn by clique-sums.*

In the first lecture, we have seen Theorem 4 follows from the following lemma, which as we now show is a consequence of Theorem 3. Recall a set $\mathcal{L}$ of separations in a graph G is a *location* if for all distinct separations $(A_1, B_1), (A_2, B_2) \in \mathcal{L}$, we have $A_1 \subseteq B_2$. The *center* of the location is the graph C obtained from $\bigcap_{(A,B)\in\mathcal{L}} B$ by adding all edges of cliques with vertex sets $V(A \cap B)$ for $(A, B) \in \mathcal{L}$.

Lemma 5. *For every graph F, there exist integers $\alpha < \theta$, k, and ρ such that the following holds. For every graph G and a tangle $\mathcal{T}$ in G of order at least θ, if $F \not\preceq G$, then there exists a location $\mathcal{L} \subseteq \mathcal{T}$ whose center is (a, k, ρ)-near-embedded in a surface in which F cannot be drawn.*

Proof. Let $\alpha < \theta$, k, p, A, Σ, and a $(\mathcal{T} - A)$-central segregation S of $G - A$ of type (k, p) with an arrangement in Σ be obtained using Theorem 3. Let $\rho = 2p + 1$. The location $\mathcal{L}$ is obtained as follows:

- For each cell $(C, \Omega) \in S$, we include the separation (A_C, B_C), where $A_C = G[V(C) \cup A]$ and $V(A_C \cap B_C) = \Omega \cup A$.
- For each p-vortex $(C, \Omega) \in S$, let (P, β) be the path decomposition from Lemma 1, where $P = v_1 v_2 \ldots v_m$. In $\mathcal{L}$, we include all separations (A_i, B_i) for $i = 1, \ldots, m$, where $A_i = G[\beta(v_i) \cup A]$ and $V(A_i \cap B_i) = A \cup \{v_i\} \cup X_i$, with $X_i = (\beta(v_i) \cap (\beta(v_{i-1}) \cup \beta(v_{i+1})))$.

The (α, k, ρ)-near-embedding of the center of $\mathcal{L}$ is obtained by making A into apex vertices, replacing each cell in the arrangement by a clique of size at most three, and replacing each p-vortex by a vortex of width at most ρ, whose bags are the sets X_i. □

2 Growing animals

Let $\mathcal{T}$ be a tangle in a graph G. For a surface Σ, $H \subseteq G$, and a tangle $\mathcal{T}_H$ of order γ in H, we say $(H, \mathcal{T}_H)$ is a Σ-*span of order* γ in $(G, \mathcal{T})$ if H is a subdivision of a 3-connected graph, H has a 2-cell drawing in Σ and $\mathcal{T}_H$ is respectful for this drawing, and $\mathcal{T}$ is *conformal* with $\mathcal{T}_H$ (i.e., the tangle induced in G by $\mathcal{T}_H$ is a subset of $\mathcal{T}$). The results from the 6th lecture imply the following.

Lemma 6. *For every graph F and a surface Σ in which F can be drawn, there exists γ such that the following claim holds. If $(G, \mathcal{T})$ contains a Σ-span of order γ, then $F \preceq G$.*

For a span $(H, \mathcal{T}_H)$, let d denote the distance function in H derived from $\mathcal{T}_H$. An H-*path* is a path in G intersecting H exactly in its endpoints. A (γ, s)-*horn* over the span is a vertex $v \in V(G) \setminus V(H)$ for which there exist s paths from v to vertices $v_1, \ldots, v_s \in V(H)$, disjoint except for their common start in v and disjoint from H except for their ends, where $d(v_i, v_j) = \gamma$ for all $i \neq j$. For $A \subseteq V(G) \setminus V(H)$, a γ-*hair avoiding* A is a vertex $z \in V(H)$ such that there exists an H-path in $G - A$ to a vertex $y \in V(H)$ with $d(z, y) = \gamma$.

A Σ-*animal* with a horns and b hairs of strength (γ, s) is a quadruple $(H, \mathcal{T}_H, A, B)$, where

- $(H, \mathcal{T}_H)$ is a Σ-span of order γ,
- A is a set of a (γ, s)-horns over the span, and

- B is a set of b γ-hairs avoiding A such that $d(x,y) = \sigma$ for distinct $x, y \in B$.

The argument used to prove Lemma 8 in the previous lecture notes gives the following.

Lemma 7. *For any surface Σ and integer m, there exist γ and s such that the following claim holds. If $(G, \mathcal{T})$ contains a Σ-animal with $\binom{m}{2}$ horns of strength (γ, s), then $K_m \preceq G$.*

Furthermore, analogously to Lemma 9, we can show that many hairs can be combined into a horn.

Lemma 8. *For any surface Σ and integers m, s, γ, a, there exist γ' and b such that the following claim holds. If $(G, \mathcal{T})$ contains a Σ-animal with a horns and b hairs of strength $(\gamma', s+1)$, then either $K_m \preceq G$, or $(G, \mathcal{T})$ contains a Σ-animal with $a+1$ horns of strength (γ, s).*

Next, let us argue that H-paths between distant vertices of H either can be used to improve the animal, or can be all disrupted by a small number of vertices. We say a Σ-span $(H, \mathcal{T}_H)$ is *δ-flat* if for every H-path, its ends u and v satisfy $d(u,v) < \delta$. A *δ-zone* around a vertex $v \in V(H)$ is an open disk $\Lambda \subset \Sigma$ bounded by a cycle C in H such that all atoms in the closure of Λ are at distance at most δ from v and conversely, all atoms at distance at most $\delta - 2$ belong to Λ. *Clearing the zone* means deleting vertices and edges of H drawn in Λ; note that the resulting graph H' contains a tangle $\mathcal{T}_{H'}$ of order $\theta - O(\delta)$ conformal with $\mathcal{T}_H$ such that the distances according to $d_{\mathcal{T}_{H'}}$ are by at most $O(\delta)$ smaller than those according to $d_{\mathcal{T}_H}$.

Lemma 9. *For any surface Σ and integers a, b, θ_1, there exist δ, α and θ_0 such that the following claim holds for every integer s. If $(G, \mathcal{T})$ contains a Σ-animal $\mathcal{A} = (H, \mathcal{T}, A, B)$ with a horns and b hairs of strength (θ, s) and $\theta \geq \theta_0$, then at least one of the following holds:*

1. *$(G, \mathcal{T})$ contains a $(\Sigma +$ handle$)$- or $(\Sigma +$ crosshandle$)$-span of order θ_1, or*

2. *$(G, \mathcal{T})$ contains a Σ-animal with a horns and $b+1$ hairs of strength (θ_1, s), or*

3. *$(G, \mathcal{T})$ contains a Σ-animal $(H', \mathcal{T}_{H'}, A, B')$ with a horns and b hairs of strength $(\theta - \delta, s)$ and a set $Z \subseteq V(G) \setminus V(H')$ of size at most α such that $A \subseteq Z$ and $(H', \mathcal{T}_{H'})$ is δ-flat in $G - Z$.*

Proof idea. For $y \in B$, choose a δ'-zone Λ_y around y bounded by a cycle C_y for $\delta' = O(\delta/b)$, and let $(H', \mathcal{T}_{H'}$ be the Σ-span of order $\theta - \delta$ obtained by clearing the zones. For each $y \in B$, choose a vertex y' of C_y joined to y by a path in Λ_y, and let $B' = \{y' : y \in B\}$. Then $(H', \mathcal{T}_{H'}, A, B')$ is an animal with a horns and b hairs of strength $(\theta - \delta, s)$. Moreover, Λ_y can be chosen so that C_y contains a set X_y that is free in $\mathcal{T}_{H'}$ and satisfies the following technical connectivity condition ($\star$): Suppose $v_1, v_2, \ldots, v_t$ is a sequence of vertices of $H \cap \Lambda_y$ with $d(y, v_1) \geq \delta - \theta_1$ and $d(v_{j-1}, v_j) < \theta_1$ for $j = 2, \ldots, t$. If $d(y, v_m) < \theta_1$, then $H \cap (\Lambda_y \cup X_y)$ contains $b^2\theta_1^2$ pairwise vertex-disjoint paths from $\{v_1, \ldots, v_m\}$ to X_y. Let us remark that to ensure this condition holds, it is in particular necessary that $\delta \gg \theta_1$.

If for some distinct $y_1, y_2 \in B$ there are at least θ_1^2 disjoint H'-paths from X_{y_1} to X_{y_2}, then we can select θ_1 of them such that the order of their ends in C_{y_1} and C_{y_2} is either the same or opposite. Adding these paths to H', we obtain a ($\Sigma +$ handle)- or ($\Sigma +$ crosshandle)-span of order θ_1 in G. Hence, we can assume this is not the case, and thus by Menger's theorem, G contains a set Z_0 of size less than $b^2\theta_1^2$ intersecting all paths with ends in X_{y_1} and X_{y_2} for distinct $y_1, y_2 \in B$.

If $Z = (Z_0 \setminus V(H')) \cup A$ intersects all H'-path with ends $u, v \in V(H')$ satisfying $d_{\mathcal{T}_{H'}}(u, v) \geq \delta$, then the last outcome holds. Hence, suppose Q is such a path avoiding Z. Let u' be the first vertex of Q after u belonging to H. If $d(u, u') \geq \theta_1$, we can add u as a hair to $\mathcal{A}$ and the second outcome holds. Hence, $d(u, u') < \theta_1$, and defining v' symmetrically, we have $d(v, v') < \theta_1$. Consequently, $d(u', v') > \delta - 2\theta_1 > 2\delta'$. Therefore, $u' \in \Lambda_{y_1}$ and $v' \in \Lambda_{y_2}$ for distinct $y_1, y_2 \in B$. By the choice of Z_0, it cannot be the case that for both $i \in \{1, 2\}$, $H \cap (\Lambda_{y_i} \cup X_{y_i})$ contains a path from X_{y_i} to $V(Q)$ disjoint from Z_0; we can assume no such path exists for $i = 1$.

Consider the segment of Q starting with u' which intersects H only in $H \cap \Lambda_{y_1}$. Let $v_1, \ldots, v_m$ be these intersections in order they appear on Q, with m chosen maximum so that $d(v_{j-1}, v_j) < \theta_1$ for $j = 2, \ldots, t$. Since $H \cap (\Lambda_{y_1} \cup X_{y_1})$ does not contain a path from X_{y_1} to $V(Q)$ disjoint from Z_0, by ($\star$) we have $d(y_1, v_m) > \theta_1$. Let w be the next intersection of Q with H after v_m. Then $d(v_m, w) \geq \theta_1$, by the maximality of m if $w \in \Lambda_{y_1}$ and since $d(y_1, y_3) \geq \theta$ and $d(w_m, y_1) \leq \delta$ if $w \in \Lambda_{y_3}$ for some $y_3 \neq y_1$. Then we can add v_m as a new hair to $\mathcal{A}$, and the second outcome holds. □

Let us now refine the last outcome. Suppose $(H, \mathcal{T}_H)$ is a Σ-span. Another Σ-span $(H', \mathcal{T}_{H'})$ is a λ-*rearrangement* of $(H, \mathcal{T}_H)$ around a vertex $v \in V(H)$ if every vertex or edge of H at distance more than λ from v belongs to H' and the distances according to $d_{\mathcal{T}_{H'}}$ are by at most $4\lambda + 2$ smaller than those according to $d_{\mathcal{T}_H}$. We say that $(H, \mathcal{T}_H)$ is (λ, δ)-*flat* if all its λ-rearrangements are δ-flat.

Lemma 10. *For any surface Σ and integers a, s, b, θ_1, there exist δ and α such that for all λ and θ_2 there exists θ for which the following claim holds. If $(G,\mathcal{T})$ contains a Σ-animal $\mathcal{A}=(H,\mathcal{T},A,B)$ with a horns and b hairs of strength $(\theta, s+\alpha)$, then at least one of the following holds:*

1. *$(G,\mathcal{T})$ contains a $(\Sigma+$ handle$)$- or $(\Sigma+$ crosshandle$)$-span of order θ_1, or*

2. *$(G,\mathcal{T})$ contains a Σ-animal with a horns and $b+1$ hairs of strength (θ_1, s), or*

3. *there is a set $Z\subseteq V(G)$ of size at most α such that $(G-Z,\mathcal{T}-Z)$ contains a (λ,δ)-flat Σ-span of order θ_2.*

Proof idea. Let δ, α, and θ_0 be as in Lemma 9, with $\delta>\theta_1$. Let $\theta=\max(\theta_0,\theta_2+\delta,2\theta_1+\delta+4\lambda+2)$. Apply Lemma 9 to $\mathcal{A}$; the first two outcomes correspond to the outcomes of this lemma, and thus we can assume $(G,\mathcal{T})$ contains a Σ-animal $(H',\mathcal{T}_{H'},A,B')$ with a horns and b hairs of strength $(\theta-\delta,s+\alpha)$ and a set $Z\subseteq V(G)\setminus V(H')$ of size at most α such that $A\subseteq Z$ and $(H',\mathcal{T}_{H'})$ is δ-flat in $G-Z$. If the Σ-span $(H',\mathcal{T}_{H'})$ is (λ,δ)-flat in $G-Z$, then the third outcome holds.

Otherwise, there exists a λ-rearrangement $(H'',\mathcal{T}_{H''})$ of $(H',\mathcal{T}_{H'})$ around a vertex w and an H''-path Q in $G-Z$ whose ends u and v satisfy $d_{\mathcal{T}_{H''}}(u,v)\ge\delta$. For a vertex $x\in A$, consider $s+1$ of the paths showing that x is a horn which are disjoint from $Z\setminus\{x\}$. At most one of them intersects $H''-H'$, as otherwise $(H',\mathcal{T}_{H'})$ would not be δ-flat in $G-Z$. Consequently, each element of A is a (θ_1,s)-horn over $(H'',\mathcal{T}_{H''})$. Furthermore, at most one of the hairs in B' is at distance less than $\theta_1+\lambda$ from w; let y be this hair, or an arbitrary hair in B' if no hair is close to w. Then $(H'',\mathcal{T}_{H''},A,(B'\setminus\{y\})\cup\{u,v\})$ is a Σ-animal with a horns and $b+1$ hairs of strength (θ_1,s), and the second outcome of the lemma holds. □

For a span $(H,\mathcal{T}_H)$, a face of H is an *eye* if there exist vertices x_1, …, x_4 appearing in order in the cycle bounding f such that there exist disjoint H-paths from x_1 to x_3 and from x_2 to x_4 and the set $\{x_1,\ldots,x_4\}$ is free in $\mathcal{T}_H$. The homework assignment for Lesson 6 implies the following result (the assumption that the span is $(\beta-10)$-flat is used to show that the crossing paths for different eyes are pairwise disjoint).

Lemma 11. *For any surface Σ and integer m, if $\beta\gg m,\Sigma$ and $(G,\mathcal{T})$ contains a $(\beta-10)$-flat Σ-span with m^4 eyes pairwise at distance at least β apart, then $K_m\preceq\mathcal{T}$.*

We now can improve or embed a flat span.

Lemma 12. *For any surface Σ and integers m, δ, and θ_1, there exist λ, θ_2, and p such that the following claim holds. If $(G, \mathcal{T})$ contains a (λ, δ)-flat Σ-span of order at least θ_2 and $K_m \not\preceq G$, then either*

1. *$(G, \mathcal{T})$ contains a (Σ + crosscap)-span of order θ_1, or*
2. *G has a $\mathcal{T}$-central segregation of type (m^4, p) with an arrangement in Σ.*

Proof idea. Choose $\gamma_{-1} \gg \lambda_0 \gg \gamma_0 \gg \lambda_1 \gg \gamma_1 \gg \ldots \gg \lambda m^4 \gg \gamma_{m^4}$, where $\gamma_{m^4} \geq \max(\delta, \beta) + 10$ for β from Lemma 11. Set $\theta_2 = \gamma_{-1}$ and $p \gg \lambda = \lambda_0$.

Let $k \in \{0, \ldots, m^4\}$ be maximum such that $(G, \mathcal{T})$ contains a (λ_k, δ)-flat Σ-span $(H, \mathcal{T}_H)$ of order at least γ_{k-1} with k eyes $f_1, \ldots, f_k$ pairwise at distance at least γ_k apart. Note that $k < m^4$, as otherwise $K_m \preceq G$ by Lemma 11.

Consider a vertex $v \in V(H)$ such that an eye can be created by cleaning a 4δ-zone around v. The maximality of k implies that the distance between v and some f_i is less than $\gamma_{k+1} + 16\delta + 2 \ll \lambda_{k+1}$. For $i = 1, \ldots, k$, let Λ_i be the corresponding zone around f_i. The "local planarity" together with rigidness of H implies that that everything outside of these zones can be broken up into cells with an arrangement in Σ. We now apply the results from the homework assignment to each zone Λ_i and all H-bridges of G that attach to it. If it does not contain a large crooked transaction, then it can be decomposed into a rural neighborhood and a p-vortex, and if this happens for all i, we obtain a desired $\mathcal{T}$-central segregation of type (m^4, p) with an arrangement in Σ.

Hence, suppose this does not happen for some i. The large crooked transaction contains one of crosscap, jump, or double-cross type. Crosscap-type crooked transaction can be used to rearrange $(H, \mathcal{T}_H)$ into a (Σ + crosscap)-span of order θ_1. Jump one would contradict the assumption that $(H, \mathcal{T}_H)$ is (λ_k, δ)-flat. The double-cross type can be used to rearrange and obtain one more distant eye, contradicting the maximality of k. □

Let us now combine all these results.

Corollary 13. *For any surface Σ and integers m, θ_1, s_1, a, b, there exists θ, s, p, α such that the following holds. If $(G, \mathcal{T})$ contains a Σ-animal with a horns and b hairs of strength (θ, s) and $K_m \not\preceq G$, then*

- *$(G, \mathcal{T})$ contains a (Σ + handle)-, (Σ + crosshandle)-, or (Σ + crosscap)-span of order θ_1, or*

- *$(G, \mathcal{T})$ contains a Σ-animal with a horns and $b+1$ hairs of strength (θ_1, s_1), or*

- *there exists a set $Z \subseteq V(G)$ of size at most α such that $G - Z$ has a $(\mathcal{T} - Z)$-central segregation of type (m^4, p) with an arrangement in Σ.*

Proof. Let δ and α be as in Lemma 10 for the given parameters (with $s = s_1$). Let λ, θ_2, and p be as in Lemma 12 for the given parameters. Let θ be obtained by Lemma 10 for this λ and θ_2, and let and $s = s_1 + \alpha$.

Applying Lemma 10 to $(G, \mathcal{T})$, the first outcomes correspond to the first two outcomes of this lemma. Hence, we can assume the third outcome, giving us a (λ, δ)-level Σ-span in $(G - Z, \mathcal{T} - Z)$ for a set Z of size at most α. We now apply Lemma 12; the first outcome corresponds to the first outcome of this lemma, while the second one corresponds to the third outcome of this lemma. □

3 Proof of the structure theorem

Let $m = |V(F)|$. We define $\theta = \theta(\Sigma, a, b)$, $s = s(\Sigma, a, b)$, $p = p(\Sigma, a, b)$ and $\alpha = \alpha(\Sigma, a, b)$ inductively so that the following conditions hold.

- If F can be drawn in Σ, then θ is equal to γ from Lemma 6 and $s = p = \alpha = 0$. Suppose from now on that F cannot be drawn in Σ.

- If $a \geq \binom{m}{2}$, then let θ and s be chosen according to Lemma 7, setting $\theta = \gamma$, and let $p = \alpha = 0$. Suppose from now on that $a < \binom{m}{2}$.

- Let $b_{\max}(\Sigma, a)$ be equal to b from Lemma 8 for the given Σ, m, $s(\Sigma, a+1, 0)$, $\theta(\Sigma, a+1, 0)$, and a. If $b \geq b_{\max}(\Sigma, a)$, then let $s = s(\Sigma, a+1, 0)+1$, $p = \alpha = 0$, and let θ be chosen as γ' from Lemma 8. From now on, suppose that $b < b_{\max}(\Sigma, a)$.

- Let θ and s be chosen according to Corollary 13, with θ_1 and s_1 maximum of the following:

  - $\theta(\Sigma', 0, 0)$ and $s(\Sigma', 0, 0)$ for $\Sigma' \in \{\Sigma+\text{handle}, \Sigma+\text{crosshandle}, \Sigma+\text{crosscap}\}$.
  - $\theta(\Sigma, a, b+1)$ and $s(\Sigma, a, b+1)$.

  We choose p and α as the maximum of values of p and α among these cases and those obtained from Corollary 13.

A straightforward inductive argument gives the following.

Corollary 14. *For any graph F, a surface Σ, and integers a and b, if $(G, \mathcal{T})$ contains a Σ-animal with a horns and b hairs of strength $(\theta(\Sigma, a, b), s(\Sigma, a, b))$ and $F \not\preceq G$, then there exists a set $Z \subseteq V(G)$ of size at most $\alpha(\Sigma, a, b)$ such that $G - Z$ has a $\mathcal{T}$-central segregation of type $(m^4, p(\Sigma, a, b))$ with an arrangement in some surface in which F cannot be drawn.*

Theorem 3 thus follows with $\alpha = \alpha(\texttt{sphere}, 0, 0)$, $k = |V(F)|^4$, $p = p(\texttt{sphere}, 0, 0)$, and θ large enough that (by the grid theorem), any graph with a tangle of order at least θ contains a wall of order $\theta(\texttt{sphere}, 0, 0)$—such a wall forms a `sphere`-animal with no horns and hairs.