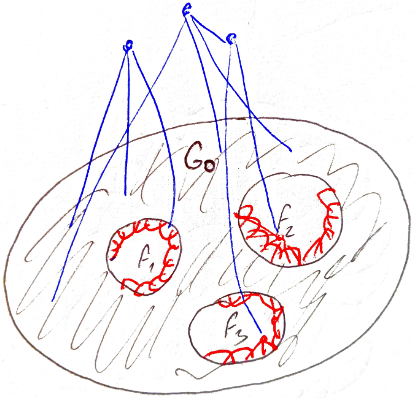
Near-embeddability

Definition

A graph G is a-near-embeddable in a surface Σ if for some graph G_0 drawn in Σ, G is obtained from an outgrowth of G_0 by at most a vortices of depth a by adding at most a apices.



Theorem (Global structure theorem)

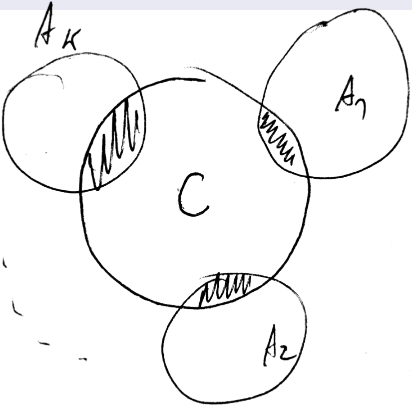
For every graph F, there exists a such that the following holds. If $F \not\preceq G$, then G has a tree decomposition such that each torso either

- *has at most a vertices, or*
- *is a-near-embeddable in some surface Σ in which F cannot be drawn.*

Definition

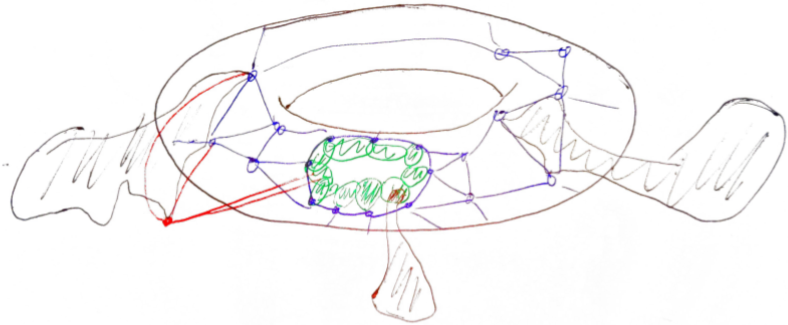
A location in G is a set of separations $\mathcal{L}$ such that for distinct $(A_1, B_1), (A_2, B_2) \in \mathcal{L}$, we have $A_1 \subseteq B_2$.

The center of the location is the graph C obtained from $\bigcap_{(A,B)\in\mathcal{L}} B$ by adding all edges of cliques with vertex sets $V(A \cap B)$ for $(A, B) \in \mathcal{L}$.

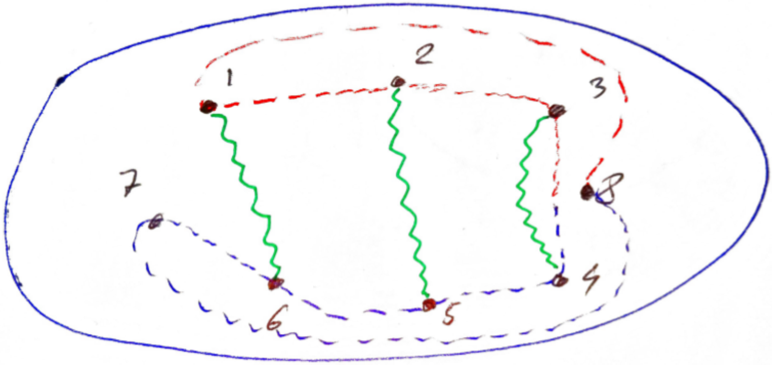


Theorem (Local structure theorem, version 1)

For every graph F, there exists a such that the following holds. If $F \not\preceq G$ and $\mathcal{T}$ is a tangle in G of order at least a, then there exists a location $\mathcal{L} \subseteq \mathcal{T}$ whose center is a-near-embeddable in some surface Σ in which F cannot be drawn.

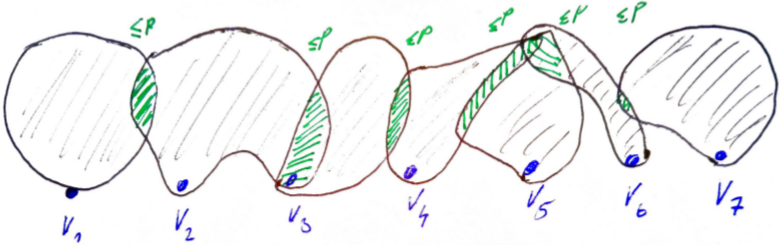


- Society: a graph G with a cyclic sequence Ω of interface vertices.
- Transaction of order p: p vertex-disjoint paths between consecutive subsequences of Ω.
- p-vortex: Society without any transaction of order p.

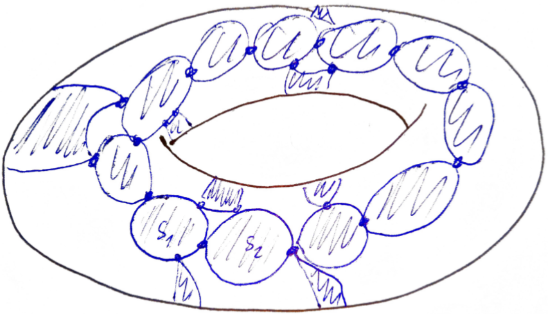


Lemma (Vortex decomposition)

If (G, Ω) is a p-vortex and $\Omega = (v_1, \ldots, v_m)$, then G has a path decomposition (P, β) over the path $P = v_1 v_2 \ldots v_m$ of adhesion at most p such that $v_i \in \beta(v_i)$ for $i = 1, \ldots, m$.

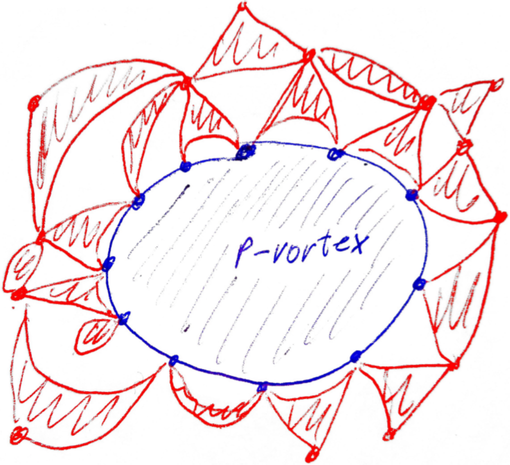


- Subsociety (S, Ω) of a graph G: S induced subgraph of G, only vertices of Ω incident with edges of $E(G) \setminus E(S)$.
- Segregation of G: Subsocieties $\{(S_i, \Omega_i) : i = 1, \ldots, n\}$ s.t.
    - $G = S_1 \cup \ldots \cup S_n$,
    - $S_i \cap S_j = \Omega_i \cap \Omega_j$ for $i \neq j$.
- Arrangement in Σ
    - Societies $\mapsto$ disks in Σ with disjoint interiors.
    - Interface vertices $\mapsto$ points in the disk boundary in a matching order.



Cell: society with at most three interface vertices. Segregation is of type (k, p) if

- all but at most k elements are cells,
- the remaining elements are p-vortices.



For a tangle $\mathcal{T}$, a segregation $\mathcal{S}$ is $\mathcal{T}$-central if there is no $(A, B) \in \mathcal{T}$ and $(S, \Omega) \in \mathcal{S}$ such that $B \subseteq S$.

Theorem (Local structure theorem, version 2)

For every graph F, there exist integers $\alpha < \theta$, k, and p such that the following holds. If $F \not\preceq G$ and $\mathcal{T}$ is a tangle in G of order at least θ, then there exists $A \subseteq V(G)$ of size at most α, a surface Σ in which F cannot be drawn, and a $(\mathcal{T} - A)$-central segregation of $G - A$ of type (k, p) with an arrangement in Σ.

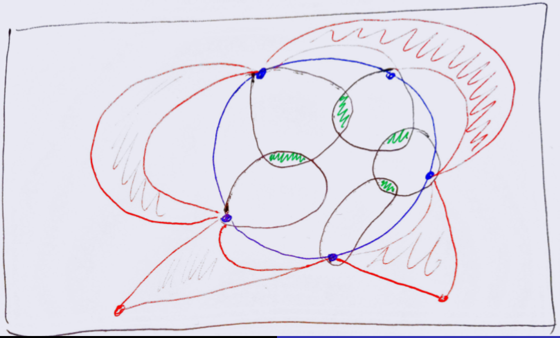
Lemma

Version 2 implies version 1.

Proof.

To obtain the location center,

- replace each cell by a clique drawn in Σ,
- apply Vortex decomposition lemma to p-vortices, bags of form $\{v_i\} \cup (\beta(v_i) \cap \beta(v_{i-1})) \cup (\beta(v_i) \cap \beta(v_{i+1}))$.



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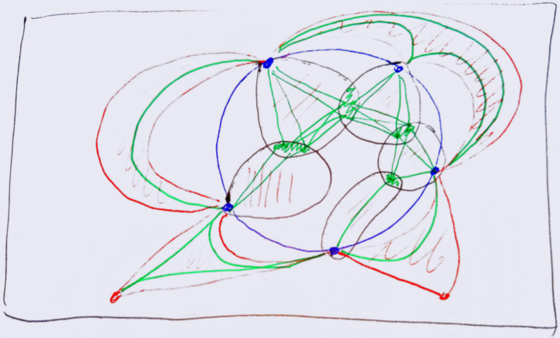
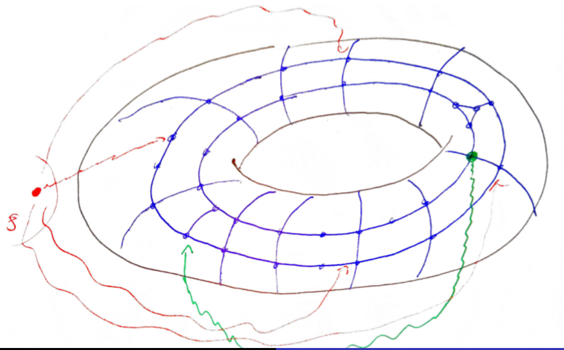
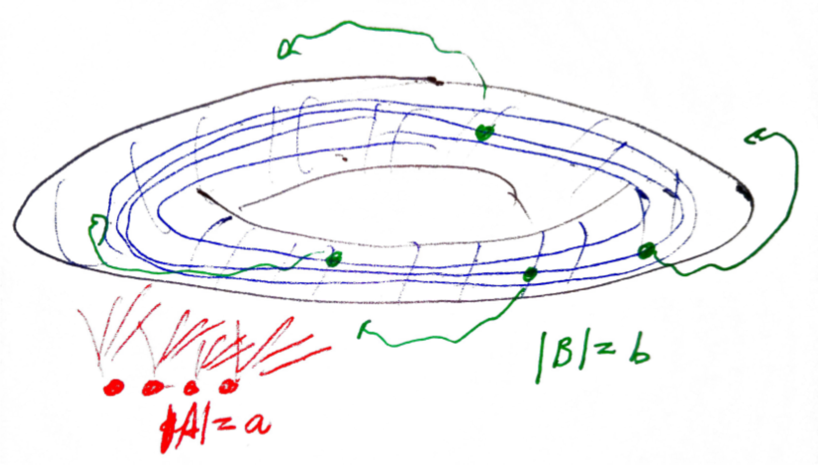


- Σ-span in $(G, \mathcal{T})$ of order θ is $(H, \mathcal{T}_H)$, where
  - $H \subseteq G$ is a subdivision of a 3-connected graph,
  - H is drawn in Σ and $\mathcal{T}_H$ is a respectful tangle of order θ conformal with $\mathcal{T}$.
- H-path: path in G intersecting H exactly in its ends. $u \in V(H)$ is an A-avoiding hair if there exists an H-path in $G - A$ whose other end u' satisfies $d_{\mathcal{T}_H}(u, u') = \theta$.
- s-horn: a vertex $v \in V(G) \setminus V(H)$, disjoint paths from v to $v_1, \ldots, v_s \in V(H)$, $d_{\mathcal{T}_H}(v_i, v_j) = \theta$ for $i \neq j$.



$(H, \mathcal{T}_H, A, B)$ is a <u>Σ-animal with a horns and b hairs of strength (θ, σ)</u> in $(G, \mathcal{T})$ if:

- $(H, \mathcal{T}_H)$ is a Σ-span in $(G, \mathcal{T})$ of order θ
- A is a set of a s-horns.
- B is a set of b A-avoiding hairs, pairwise at distance θ from one another.



Theorem

F drawn in Σ, $(G, \mathcal{T})$ contains a Σ-span of sufficiently large order $\Rightarrow F \preceq G$.

Lemma (Horn Lemma)

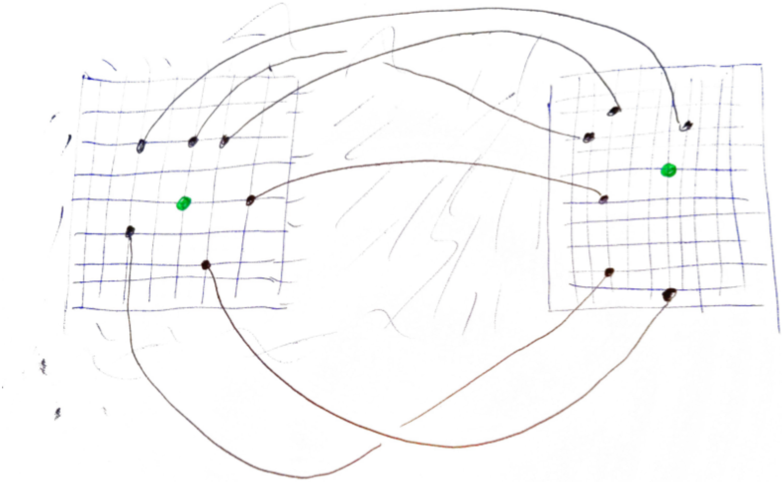
If $(G, \mathcal{T})$ contains a sufficiently strong Σ-animal with $\binom{m}{2}$ horns, then $K_m \preceq G$.

Lemma (Hairs-to-horn Lemma)

If $(G, \mathcal{T})$ contains a sufficiently strong Σ-animal with a horns and sufficiently many hairs, then either $K_m \preceq G$ or $(G, \mathcal{T})$ also contains a Σ-animal with $a + 1$ horns of strength (θ, s).

- Large treewidth $\Rightarrow$ large wall $=$ strong sphere-animal with no horns or hairs.
- Repeatedly, unless $F \preceq G$ or the decomposition is found:
  - find a slightly weaker animal in higher-genus surface, or
  - find a slightly weaker animal with one more hair.
- Genus at least the genus of F: $F \preceq G$.
- Many hairs accumulate: Hairs-to-horn Lemma gives $F \preceq G$ or one more horn.
- Many horns accumulate: Horn Lemma gives $F \preceq G$.

- No hair can be added $\Rightarrow$ all long jumps start and end near elements of B.
- Many disjoint long jumps $\Rightarrow$ $(\Sigma + \text{handle})-$ or $(\Sigma + \text{crosshandle})-$span of large order.



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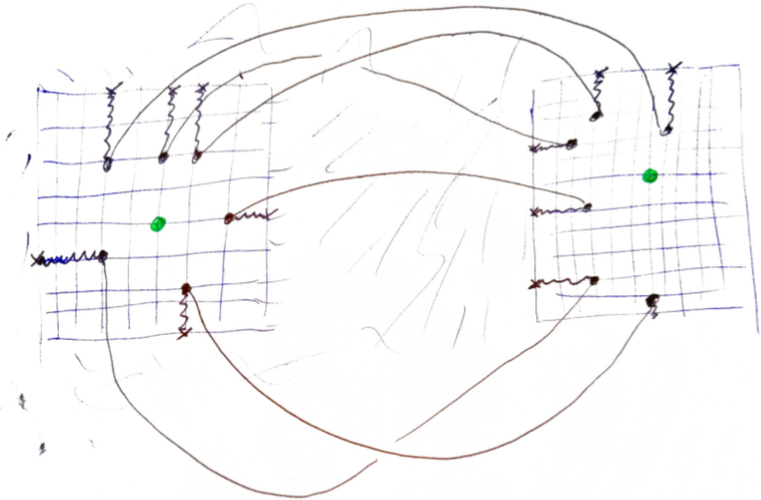

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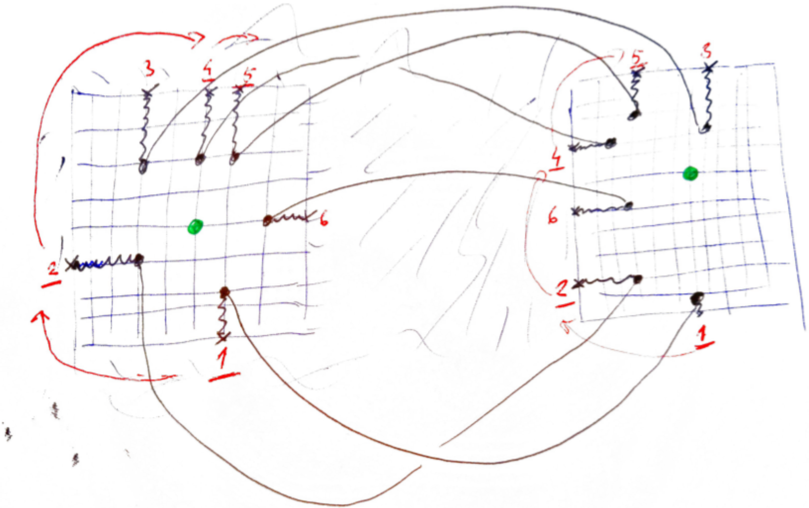

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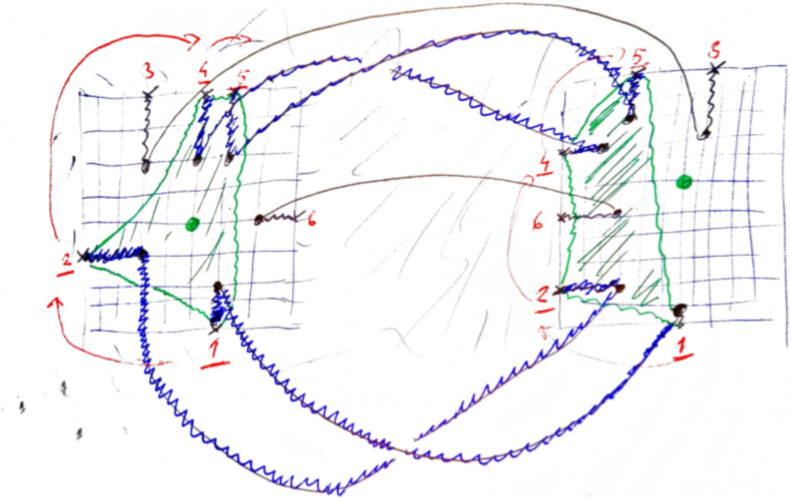
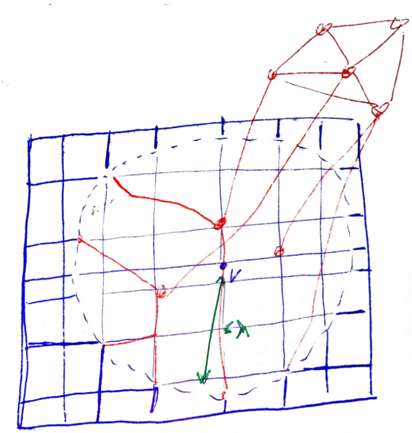


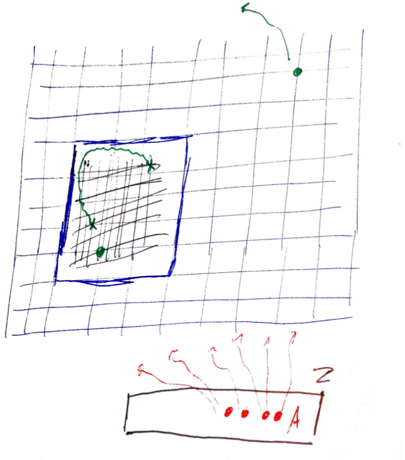
- Otherwise, all can be interrupted by a small set $Z \supseteq A$ (Menger).
- In $G - Z$, $(H, \mathcal{T}_H)$ is flat: no long jumps.

$(H', \mathcal{T}_{H'})$ is a rearrangement of $(H, \mathcal{T}_H)$ within λ of $v \in V(H)$ if

- atoms of $H - H'$ are at distance at most λ from v,
- distances according to $\mathcal{T}_{H'}$ are by at most $(4\lambda + 2)$ smaller.



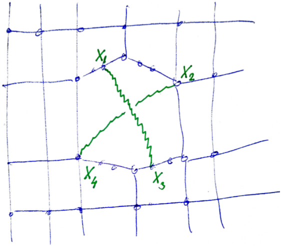
- Long jump after a local rearrangement $\Rightarrow$ one more hair.



- We can assume $(H, \mathcal{T}_H)$ is flat in $G - Z$ even after a local rearrangement.

Face f is an eye in $(H, \mathcal{T}_H)$ if

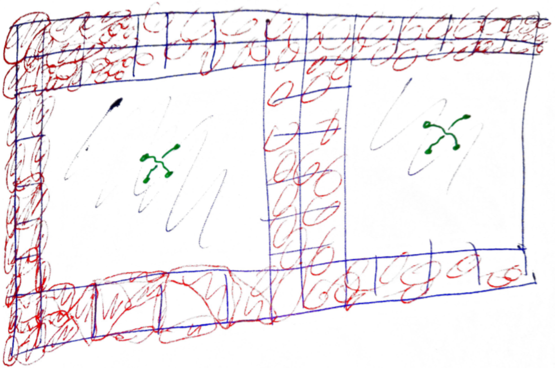
- $x_1, \ldots, x_4$ in the boundary of H in order, $\{x_1, \ldots, x_4\}$ free in $\mathcal{T}_H$,
- disjoint H-paths from x_1 to x_3 and x_2 to x_4.



Lemma (Cross Lemma)

$(H, \mathcal{T}_H)$ is flat and contains m^4 pairwise distant eyes $\Rightarrow K_m \preceq G$.

- Locally rearrange $(H, \mathcal{T}_H)$ to obtain maximum number of distant eyes $f_1, \ldots, f_k$, $k < m^4$.
- Far from the eyes: Impossible to rearrange non-planarly $\Rightarrow$ segregation into cells with arrangement in Σ.



Around each eye:

- No large crooked transaction: *p*-vortex + rural neighborhood.
- Crosscap transaction $\rightarrow$ $(\Sigma +$ crosscap)-span.
- Jump transaction $\rightarrow$ non-flat after rearrangement.
- Double-cross transaction $\rightarrow$ more distant eyes.