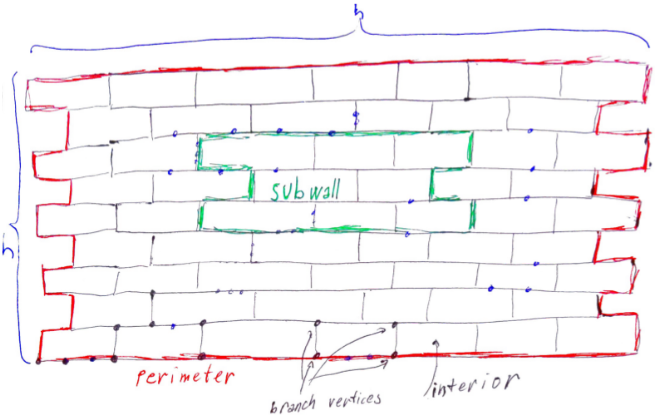
A wall of height *h*:



Theorem (A reformulation of the grid theorem)

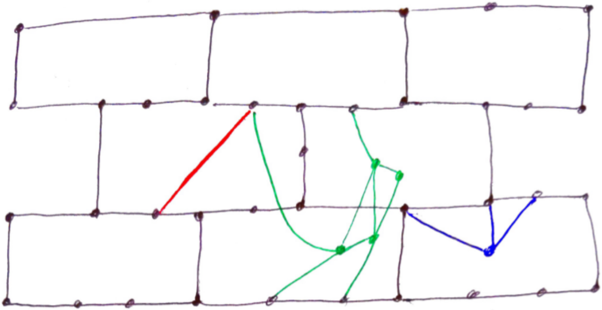
For every h, there exists t such that every graph of treewidth at least t contains a wall of height h as a subgraph.

Proof.

Grid minor $\Rightarrow$ unsubdivided wall minor $\Rightarrow$ wall subgraph. □

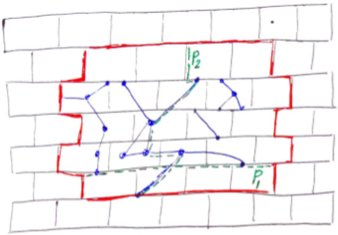
A *W*-bridge of G is

- an edge of $E(G) \setminus E(W)$ with both ends in W, or
- a component of $G - V(W)$ together with the edges to W.

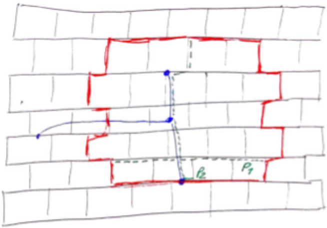


- The compass $C(W)$ of W: the perimeter K of W + the K-bridge containing the interior of W.
- A subwall W of a wall Z is dividing if $K(W) \cap Z = W$.
- A cross over W: Disjoint paths $P_1, P_2 \subset C(W)$ joining branch vertices of K s.t.
  - the ends of P_1 are in different components of $K - V(P_2)$, and
  - $(P_1 \cup P_2) \cap Z \subset W$.

dividing:



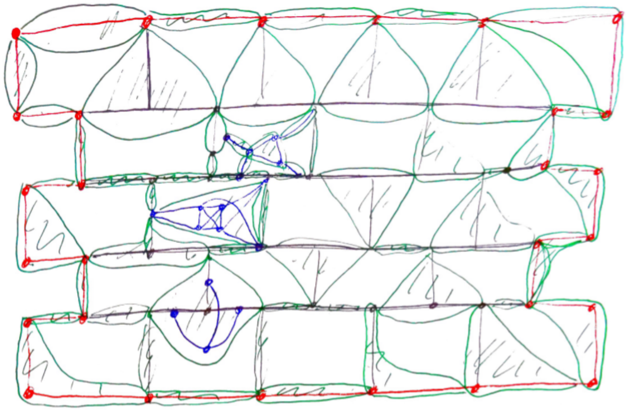
non-dividing:



Definition

A wall W is flat if there is no cross over W.

Compasses of flat walls are “almost planar”, see homework:



Theorem (The Flat Wall Theorem)

For every h and m, there exists t such that for every graph G of treewidth at least t, either

- *G contains K_m as a minor, or*
- *there exists a set X of less than $\binom{m}{2}$ vertices and a flat wall of height h in $G - X$.*

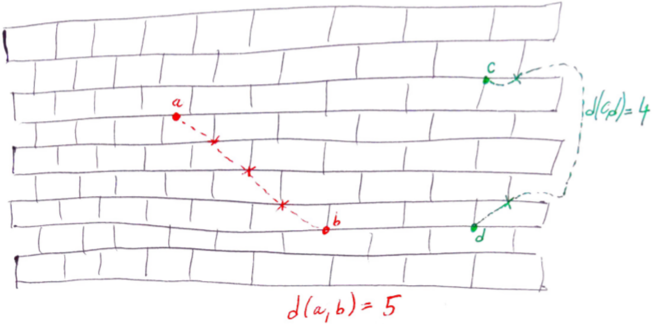
Application: Testing the presence of a fixed graph H as a minor.

For $m = |V(H)|$:

- A minor of $H \subseteq K_m$ in G, or
- small treewidth, or
- a large flat wall after removal of $< \binom{m}{2}$ vertices.

Claim: In the flat wall, one can find a vertex v such that $H \preceq G$ if and only if $H \preceq G - v$.

For $u, v \in V(W)$, let $d(u, v) =$ the minimum number of intersections of a closed curve from u to v with W.



Observation

Let $\mathcal{T}$ consist of separations (A, B) of order at most $h/2$ where A does not contain any row of W. Then $\mathcal{T}$ is a respectful tangle and $d(u, v) = \Theta(d_{\mathcal{T}}(u, v))$.

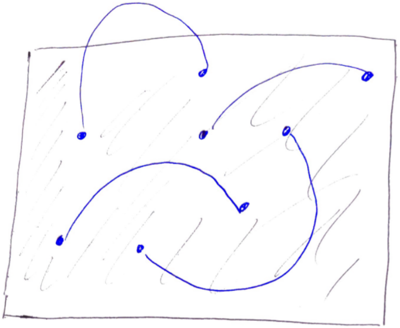
A W-path intersects W exactly in its ends.

Lemma (Jump Lemma)

$(\forall m)(\exists d_m)$: $\binom{m}{2}$ disjoint W-paths with ends in $Y \subset V(W)$,

$$d(y_1, y_2) \geq d_m$$

for all $y_1, y_2 \in Y \Rightarrow K_m \preceq G$.

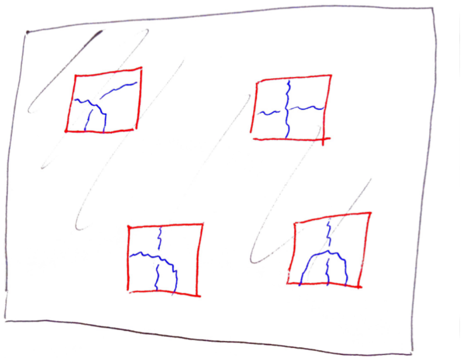


Lemma (Cross Lemma)

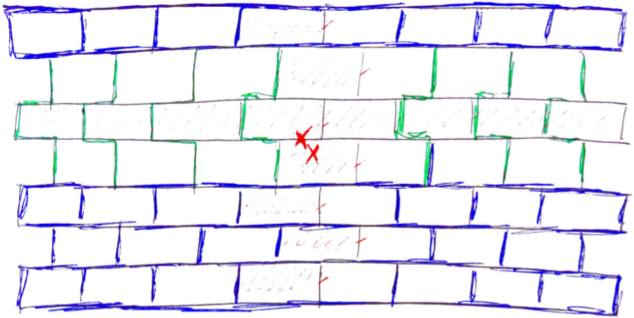
$(\forall m)(\exists d'_m)$: subwalls $W_1, \ldots, W_{m^4}$ such that

$$d(W_i, W_j) \geq d'_m$$

for $i \neq j$, disjoint crosses over all the subwalls $\Rightarrow K_m \preceq G$.



For $X \subseteq V(W)$, let W/X be obtained by removing rows and columns intersecting X.



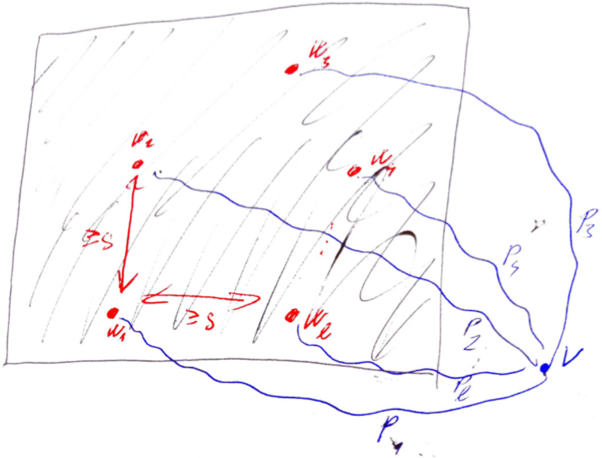
Observation

The wall W/X has height at least $h - 2|X|$,

$$d_{W/X}(u, v) \geq d(u, v) - 4|X|.$$

A vertex v is (l, s)-central over W if there exist paths $P_1, \ldots, P_l$ with ends v and $w_1, \ldots, w_l \in V(W)$ s.t.

- $P_i \cap P_j = v$ and $d(w_i, w_j) \geq s$ for $i \neq j$, and
- $P_i \cap W \subseteq \{v, w_i\}$.



Lemma (Horn Lemma)

For every m, there exist l and s such that if at least $\binom{m}{2}$ vertices are (l, s)-central over W, then $K_m \preceq G$.

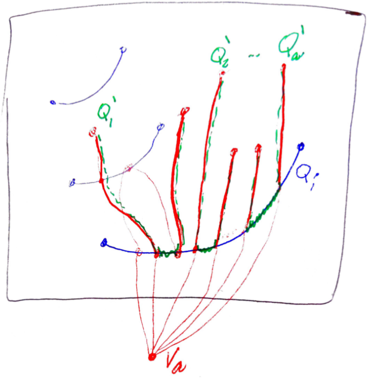
Suppose $v_1, \ldots, v_{\binom{m}{2}}$ are (l, s)-central.

- WLOG $v_1, \ldots \notin V(W)$: Consider $W/\{v_1, \ldots\}$.
- For $a = 0, \ldots, \binom{m}{2}$:
  - find a disjoint W-paths with ends $s/2$ apart and disjoint from $v_{a+1}, \ldots$
- Obtain $K_m \preceq G$ by the Jump Lemma.

Assume

- we have $Q_1, \dots, Q_{a-1}$,
- $P_1, \dots, P_{l-\binom{m}{2}}$ from centrality of v_a and disjoint from $\{v_1, \dots\}$.

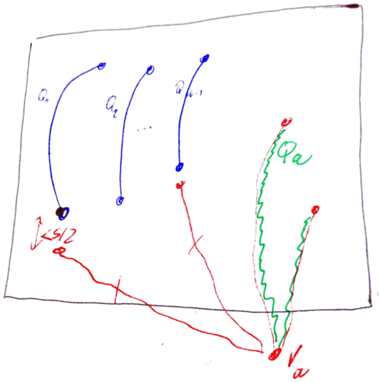
If $2a$ of $P_1, \dots,$ intersect some Q_i:



Assume

- we have $Q_1, \ldots, Q_{a-1}$,
- $P_1, \ldots, P_{l-\binom{m}{2}}$ from centrality of v_a and disjoint from $\{v_1, \ldots\}$.

If $2a$ of $P_1, \ldots$ are disjoint from $Q_1, \ldots, Q_{a-1}$:



Lemma (Non-division Lemma)

$(\forall m, l, s)(\exists k, d''_m)$: *Non-dividing subwalls* $W_1, \ldots, W_k$ *such that*

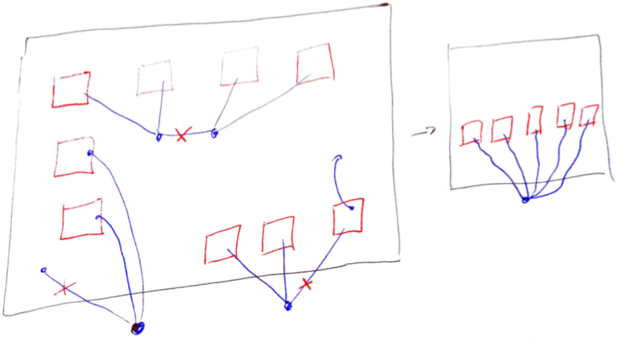
$$d(W_i, W_j) \geq d''_m$$

for $i \neq j \Rightarrow K_m \preceq G$ *or* G *contains an* (l, s)*-central vertex.*

- F = minimal subgraph of $G - E(W)$ showing $W_1, \ldots, W_k$ are non-dividing.
- F' a W-bridge of F: F' is a tree, $|F' \cap W| \geq 2$.
- W_i is solitary if only one W-bridge of F intersects W_i.

If $|F' \cap W| \geq 3$:

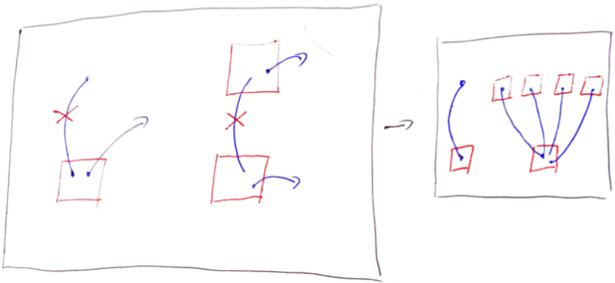
- Each leaf in a different solitary subwall.
- Subdivision of a star.



- F = minimal subgraph of $G - E(W)$ showing $W_1, \ldots, W_k$ are non-dividing.
- F' a W-bridge of F: F' is a tree, $|F' \cap W| \geq 2$.
- W_i is solitary if only one W-bridge of F intersects W_i.

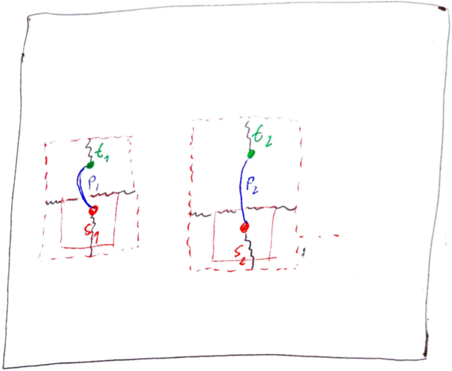
If $|F' \cap W| = 2$:

- At least one end in a solitary subwall.



- If $\Delta(F) \geq l$, then G contains an (l, s)-central vertex.
- Otherwise, F has $a \geq k/l^2$ disjoint bridges:
  - disjoint W-paths $P_1, \ldots, P_a$ with ends s_i and t_i
  - $d(s_i, s_j) \geq d''_m$ for $i \neq j$.

Case 1: $d(s_i, t_i) \leq d''_m/100$ for m^4 values of i. Apply the Cross Lemma to obtain $K_m \preceq G$:



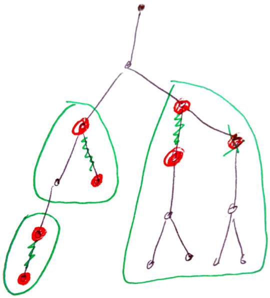
We can assume $d(s_i, t_i) > 100 d_m$ for all i.

Case 2: There exists i_0 such that $d(t_i, t_{i_0}) < 2d_m$ for $3\binom{m}{3}$ values of i.

- Let X be vertices of W at distance less than $2d_m$ from t_{i_0}.
- Apply the Jump Lemma in W/X.

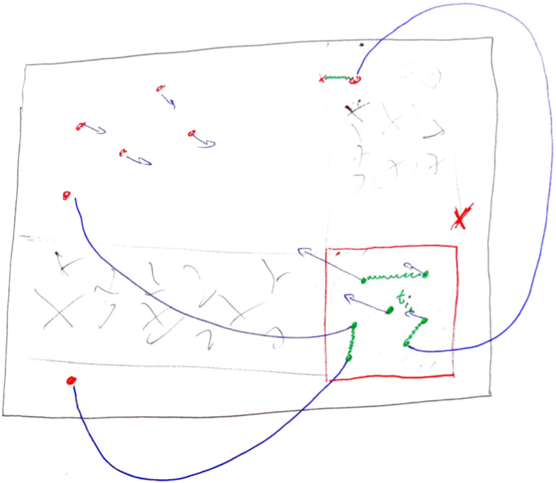
Observation

$\Delta(W[X]) \leq 3 \Rightarrow$ *many vertices t_i can be joined by disjoint paths in $W[X]$.*



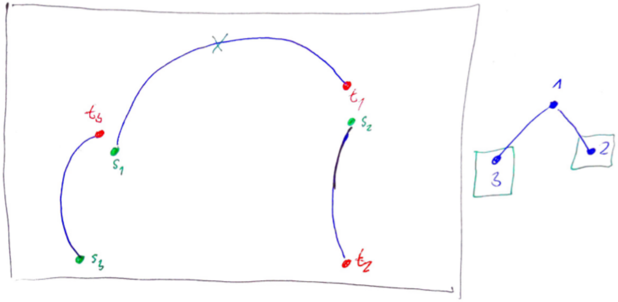
Case 2: There exists i_0 such that $d(t_i, t_{i_0}) < 2d_m$ for $3\binom{m}{3}$ values of i.

- Let X be vertices of W at distance less than $2d_m$ from t_{i_0}.
- Apply the Jump Lemma in W/X.



Case 3: At least $\frac{a}{3\binom{m}{3}}$ indices I such that $d(t_i, t_j) \geq 2d_m$ for distinct $i, j \in I$.

- Auxiliary graph H with $V(H) = I$, $ij \in E(H)$ if $d(s_i, t_j) < d_m$ or $d(s_j, t_i) < d_m$.
- $\Delta(H) \leq 2$, $\alpha(H) \geq |H|/3$.



The Jump Lemma gives $K_m \preceq G$.

Lemma (Non-division Lemma)

$(\forall m, l, s)(\exists k, d''_m)$: Non-dividing subwalls $W_1, \ldots, W_k$ such that

$$d(W_i, W_j) \geq d''_m$$

for $i \neq j \Rightarrow K_m \preceq G$ or G contains an (l, s)-central vertex.

Iteration + Horn Lemma:

Corollary

$(\forall m)(\exists k_0, d''_m)$: Subwalls $W_1, \ldots, W_k$ such that

$$d(W_i, W_j) \geq d''_m$$

for $i \neq j \Rightarrow$

- *$K_m \preceq G$ or*
- *$X \subseteq V(G), |X| < \binom{m}{2}$ such that all but k_0 of the subwalls are dividing in $(G - X) \cup W$.*

Proof of the Flat Wall Theorem:

- large treewidth $\Rightarrow$ large wall W
- many distant subwalls
- $X \subseteq V(G), |X| < \binom{m}{2}$ and many distant dividing walls in $(G - X) \cup W$
- many distant dividing walls in $G - X$
- Cross Lemma: less than m^4 of them are crossed.