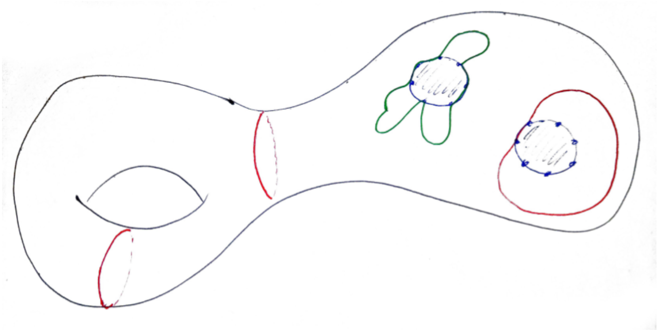
In a surface Σ, a normal drawing of G is *p-generic* if

- curves between distinct cuffs intersect G at least p times
- simple closed G-normal non-contractible curve c intersects G in $< p$ points $\Rightarrow$ for a cuff k homotopic to k, $G \cap k \subseteq G \cap c$.



We have:

Theorem

$(\forall \Sigma, k)(\exists p)$: *Let G be a graph with a normal drawing in a surface Σ with at least two holes, at most k vertices in the boundary of Σ, each cuff contains at least one vertex. Normal root assignment r is topologically feasible and the drawing of G is p-generic $\Rightarrow$ edgeless minor rooted in r.*

Want:

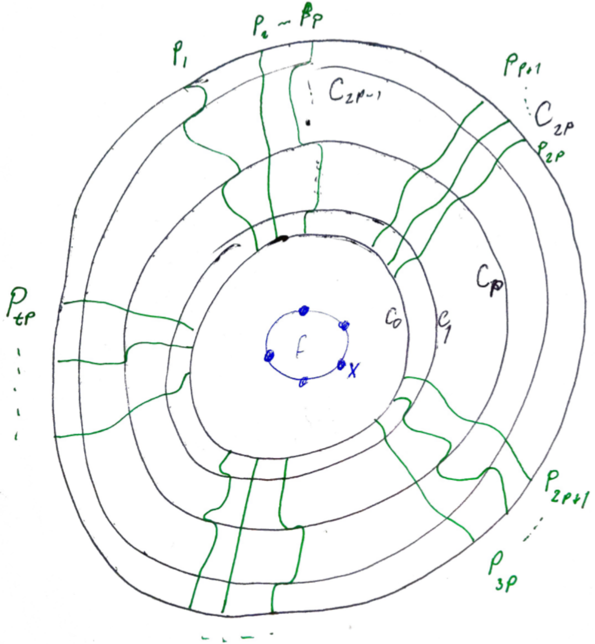
- Get rid of "at least two holes", "each cuff contains a vertex".
- Weaken the p-generic assumption: For a curve c surrounding a cuff k, only require $|G \cap c| \geq |G \cap k|$.
- Formulate in terms of respectful tangles, distance.

G with 2-cell drawing in Σ, f face, X set of t vertices in boundary of f, $\mathcal{T}$ respectful tangle.

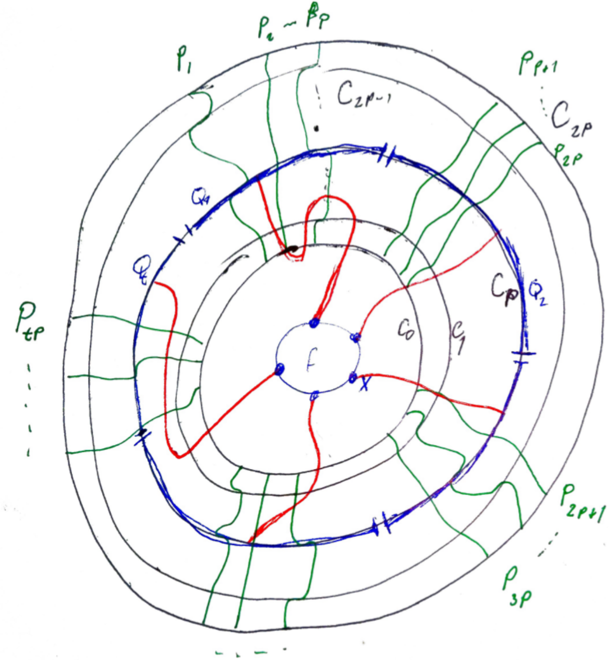
Definition

We say $(\mathcal{C}, \mathcal{P})$ is a <u>sleeve around (f, X) of order p</u> if

1. $\mathcal{C} = C_0, \ldots, C_{2p}$ disjoint cycles, $\mathcal{P} = P_1, \ldots, P_{tp}$ disjoint paths in G from C_0 to C_{2p}
2. a disk $\Delta \subseteq \Sigma$ containing f, $\mathcal{C}$, and $\mathcal{P}$ such that $d_{\mathcal{T}}(f, a) = O(tp)$ for all $a \in A(G) \cap \Delta$,
3. for any $i < j$, C_i separates f from C_j,
4. every atom a such that $d_{\mathcal{T}}(f, a) \leq tp$ is drawn between f and C_{2p},
5. for any i and j, $C_i \cap P_j$ is a connected path,
6. disjoint paths $Q_1, \ldots, Q_t \subset C_p$, each containing $P \cap C_p$ for p paths $P \in \mathcal{P}$, and p disjoint paths from X to $Q_1, \ldots, Q_t$.



P_1
$P_2 \sim P_p$
C_{2p-1}
P_{p+1}
C_{2p}
P_{2p}
C_p
C_0
C_1
F
X
P_{tp}
P_{2p+1}
P_{3p}



P_1
$P_2 \sim P_p$
C_{2p-1}
P_{p+1}
C_{2p}
P_{2p}
Q_1
Q_t
C_p
Q_2
C_0
C_1
F
X
P_{tp}
P_{2p+1}
P_{3p}

Recall: X is free in tangle $\mathcal{T}$ if for all $(A, B) \in \mathcal{T}$, if $X \subseteq V(A)$, then $|V(A \cap B)| \geq |X|$.

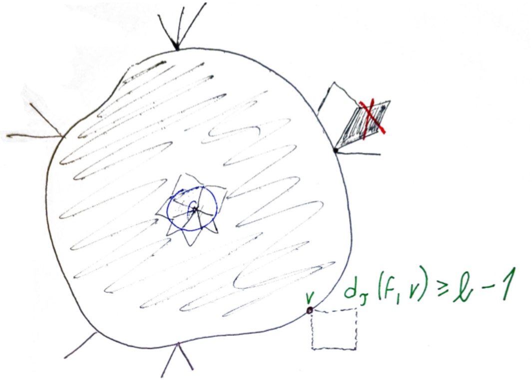
Lemma

The order θ of $\mathcal{T} \gg t$ and p, the set X is free $\Rightarrow$ a sleeve of order p around (f, X).

Recall: For $l < \theta$,

$$\bigcup_{a \in A(G), d_{\mathcal{T}}(f,a) \leq l} R(a)$$

is simply-connected.



Disk Δ for $l = O(tp)$:

- Every atom in Δ is at distance at most l from f.
- Every atom at distance at most $l - 2$ from f is drawn in Δ.

1. $\mathcal{C} = C_0, \ldots, C_{2p}$ disjoint cycles, $\mathcal{P} = P_1, \ldots, P_{tp}$ disjoint paths in G from C_0 to C_{2p}
2. a disk $\Delta \subseteq \Sigma$ containing f, $\mathcal{C}$, and $\mathcal{P}$ such that $d_{\mathcal{T}}(f, a) = O(tp)$ for all $a \in A(G) \cap \Delta$,
3. for any $i < j$, C_i separates f from C_j,
4. every atom a such that $d_{\mathcal{T}}(f, a) \leq tp$ is drawn between f and C_{2p},
5. for any i and j, $C_i \cap P_j$ is a connected path,
6. disjoint paths $Q_1, \ldots, Q_t \subset C_p$, each containing $P \cap C_p$ for p paths $P \in \mathcal{P}$, and p disjoint paths from X to $Q_1, \ldots, Q_t$.

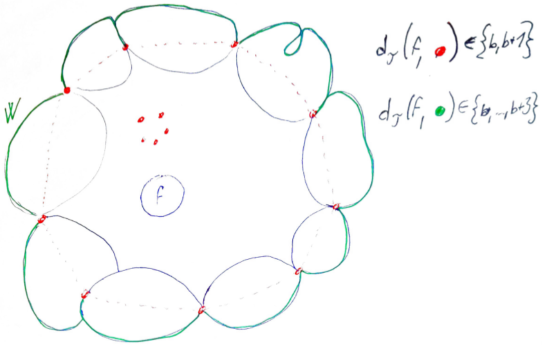
Observation

If $uv \in E(G)$, then $d_{\mathcal{T}}(u, v) \leq 2$.

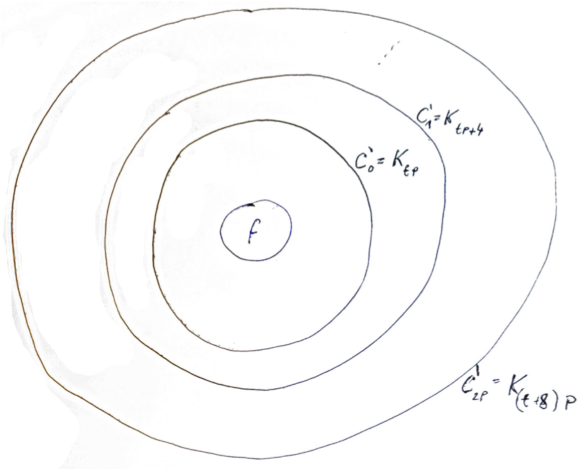
Corollary

For every $b < l$, every path from f to the boundary of Δ contains a vertex v such that $d_{\mathcal{T}}(f, v) \in \{b, b+1\}$.

- Vertices at distance b or $b+1$ separate f from the boundary of Δ.
- Closed walk W around f on distance $\{b, b+1, b+2, b+3\}$ vertices.
- Cycle K_b with $V(K_b) \subseteq V(W)$.
- v separated by K_b from $f \Rightarrow d_{\mathcal{T}}(v, f) > b$.

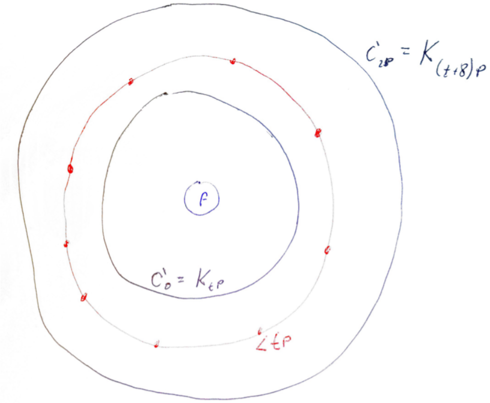


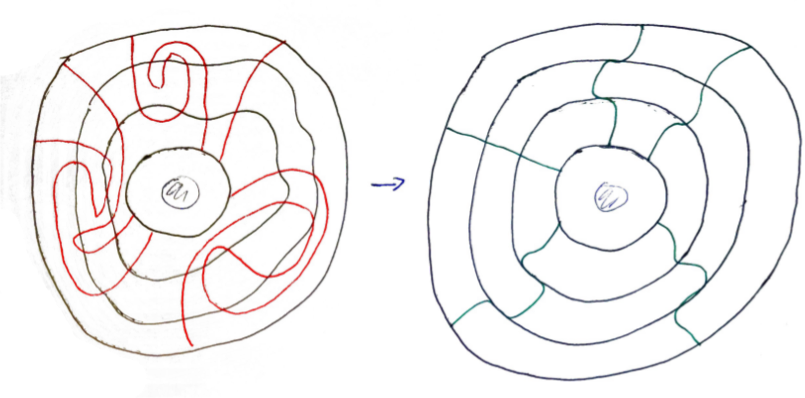
For $i = 0, 1, \ldots, 2p$, let $C'_i = K_{tp+4i}$.



G contains tp disjoint paths from C'_0 to C'_{2p}:

- Otherwise, by Menger: Curve c separating C'_0 from C'_{2p} intersecting G in less than tp vertices.
- $d_{\mathcal{T}}(f, C'_0) \geq tp$: $f \not\subset \mathrm{ins}_{\mathcal{T}}(c)$
- $\exists e : d_{\mathcal{T}}(f, e) = \theta > d_{\mathcal{T}}(f, C'_{2p}) + tp$: $C'_{2p} \not\subset \mathrm{ins}_{\mathcal{T}}(c)$.



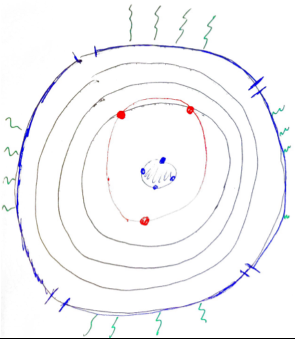


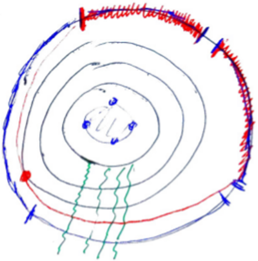
Nicely intersecting $\mathcal{C}$ and $\mathcal{P}$, with $C_0 = C'_0$ and $C_{2p} = C'_{2p}$.

1. $\mathcal{C} = C_0, \ldots, C_{2p}$ disjoint cycles, $\mathcal{P} = P_1, \ldots, P_{tp}$ disjoint paths in G from C_0 to C_{2p}
2. a disk $\Delta \subseteq \Sigma$ containing f, $\mathcal{C}$, and $\mathcal{P}$ such that $d_{\mathcal{T}}(f, a) = O(tp)$ for all $a \in A(G) \cap \Delta$,
3. for any $i < j$, C_i separates f from C_j,
4. every atom a such that $d_{\mathcal{T}}(f, a) \leq tp$ is drawn between f and C_{2p},
5. for any i and j, $C_i \cap P_j$ is a connected path,
6. disjoint paths $Q_1, \ldots, Q_t \subset C_p$, each containing $P \cap C_p$ for p paths $P \in \mathcal{P}$, and p disjoint paths from X to $Q_1, \ldots, Q_t$.

Choose $Q_1, \ldots, Q_t$ arbitrarily; t paths from X exist. Contract Q_i to q_i:

- Otherwise, by Menger: Curve c separating X from $\{q_1, \ldots, q_t\}$, intersecting G in less than t vertices.
- If $c \cap \{q_1, \ldots, q_t\} = \emptyset$:
  - $X \subset \mathrm{ins}_{\mathcal{T}}(c) \Rightarrow X$ not free.
  - $\exists e : d_{\mathcal{T}}(f, e) = \theta > d_{\mathcal{T}}(f, C_{2p}) + t$: $C_{2p} \not\subset \mathrm{ins}_{\mathcal{T}}(c)$.
- Otherwise, c intersects either $C_0, \ldots, C_{p-1}$, or p paths from $\mathcal{P}$ ending in $q_i \notin c$.





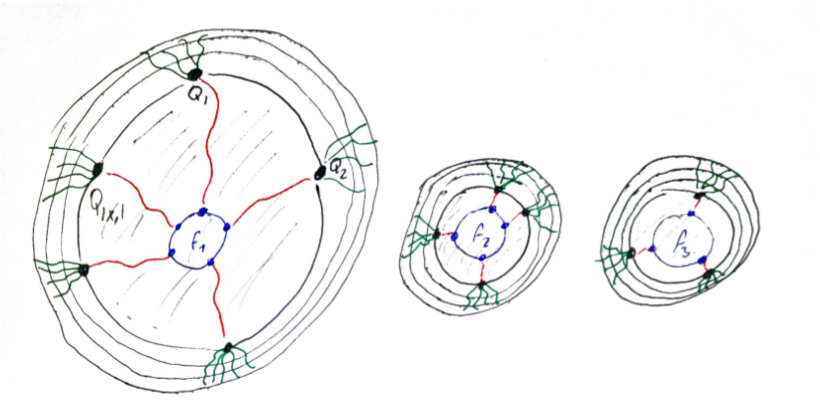
Theorem

$\forall \Sigma, k \exists \theta$: G 2-cell drawing in Σ, $\mathcal{T}$ respectful tangle of order θ, $f_1, \ldots, f_q$ faces of G, X a set of k vertices incident with them (X_i incident with f_i), H edgeless, $r : V(H) \to 2^X$ assignment of non-empty sets of roots. If

- *r is topologically feasible in $\Sigma - (f_1 \cup \ldots \cup f_q)$,*
- *$d_{\mathcal{T}}(f_i, f_j) = \theta$ for all distinct i and j, and*
- *X_i is free for $i = 1, \ldots, q$,*

then H is a minor of G rooted in r.

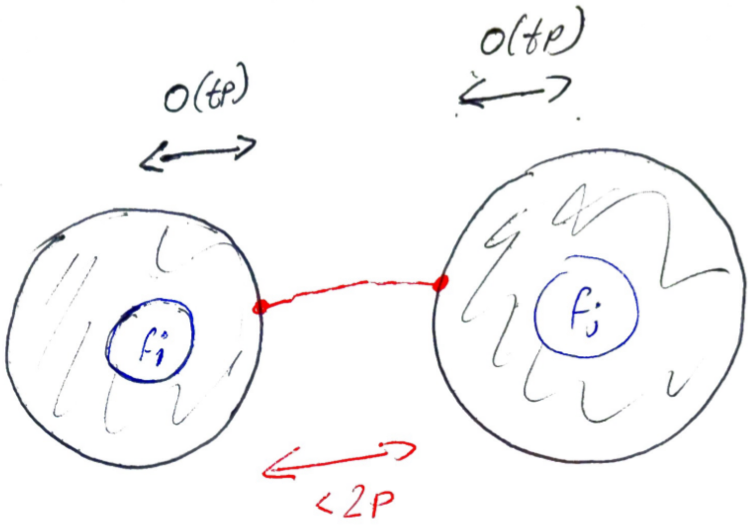
- Ensure $q \geq 2$ (choosing new faces), $X_i \neq \emptyset$ (adding roots).
- Find sleeves around $f_1, \dots, f_q$.
- In each sleeve, cut hole up to C_p, contract $Q_1, \dots, Q_{|X_i|}$.
- Apply the Theorem from the last lecture.



Need to argue: *p*-generic.

c curve between different cuffs, less than p intersections:

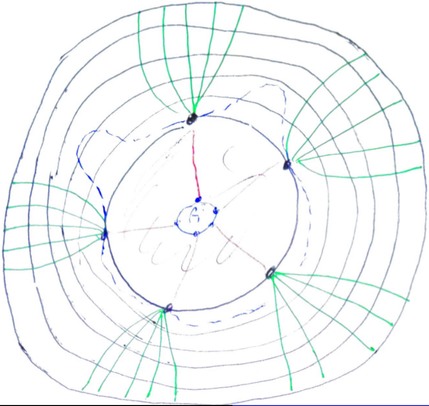
$$d_T(f_i, f_j) \le O(tp) + 2p < \theta.$$



c simple closed non-contractible curve intersecting G in less than p points:

- c touches a cuff: Cannot reach beyond C_{2p} without intersecting $C_{p+1}, \ldots, C_{2p}$.
- Otherwise: $(\forall i \neq j) d_{\mathcal{T}}(f_i, f_j) = \theta$ implies $\mathrm{ins}_{\mathcal{T}}(c)$ contains only one cuff, $d_{\mathcal{T}}(f_i, c) < p \Rightarrow c$ is between C_p and C_{2p}.

Less than p intersections with $P_1, \ldots, P_{|X_i|p} \Rightarrow X_i \subset c$.

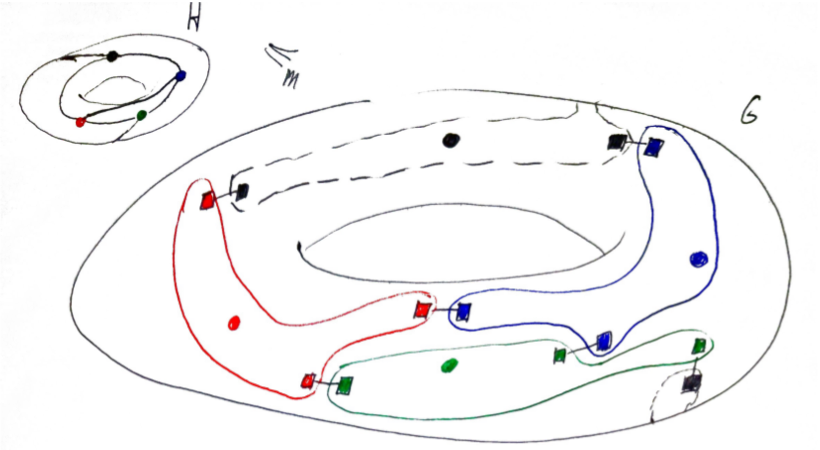


Application:

Corollary

$\forall \Sigma, H$ drawn in $\Sigma \exists \theta_1$: G 2-connected, a 2-cell drawing in Σ, $\mathcal{T}$ respectful tangle in G of order θ_1, $r(x) = \{v_x\}$ for all $x \in V(H)$. If $d_{\mathcal{T}}(v_x, v_y) = \theta_1$ for all distinct $x, y \in V(H)$, then G contains H as a minor rooted in r.

- There exist edges e, e' such that $d_{\mathcal{T}}(e, e') = \theta_1$.
- $|V(H)| + |E(H)|$ edges on a path from e to e' at distance $\Omega(\theta_1/(|V(H)| + |E(H)|))$ from one another.
- $h \in E(H) \mapsto e_h \in E(G)$ at distance θ from other $e_{h'}$, vertices $v_x : x \in V(H)$.
- G 2-connected: $e_h = \{u, v\}$ is free.



For a surface Σ with holes, integer k:

Algorithm

Input: G drawn normally in Σ, edgeless graph H with a normal root function r, at most k root vertices in total.
Output: A minor of H in G rooted in r, or decides it does not exist.

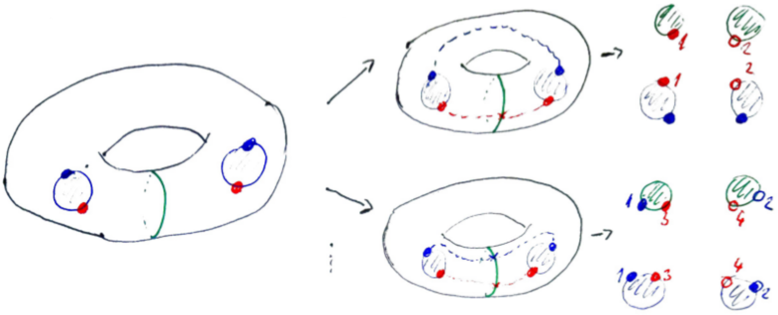
Inductively assume we have such algorithm for

- smaller genus,
- the same genus, fewer holes,
- Σ, less than k roots.

For cylinder: homework assignment.

Cutting argument ($|c \cap G|$ bounded by a function of Σ, k):

- Try all intersections of the minor with $c \cap G$.
- Cut along c, add new roots.
- Solve inductively.



To obtain H as a rooted minor by the theorem, we need

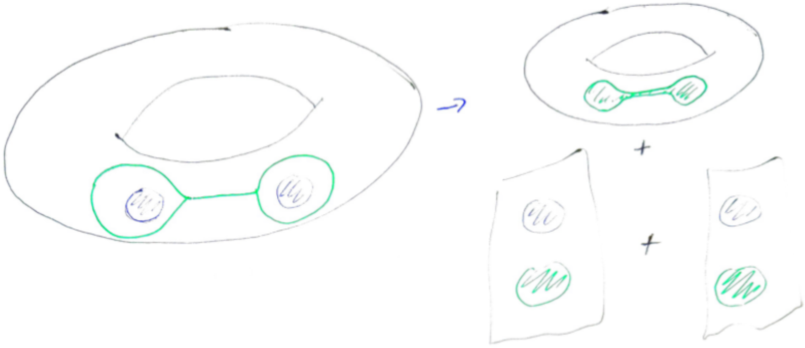
- r topologically feasible
- respectful tangle $\mathcal{T}$ of large order
- faces with roots are far in $d_{\mathcal{T}}$
- roots in a face are free.

No respectful tangle of large order.

- Σ is sphere with holes:
  - No tangle of large order $\Rightarrow$ bounded treewidth.
  - Rooted minor containment is expressible in MSOL.
- Σ has positive genus:
  - Bounded representativity.
  - Cut along non-contractible curve to decrease genus.

$d_{\mathcal{T}}(f_1, f_2)$ is small:

- Cut along a tie around f_1 and f_2.
- Decreased number of holes + cylinders.



X_1 is not free:

- Curve c around f_1 with $|G \cap c| < |X_1|$.
- Cut around c: decreased number of roots + cylinder.