In the last lesson, we gave the following sufficient condition for the existence of a rooted edgeless minor in an embedded graph. For an integer p, we say a drawing of a graph G is *p-generic* if

- every G-normal curve with ends in different cuffs intersects G in at least p points, and
- if a simple closed G-normal non-contractible curve c intersects G in less than p points, then there exists a cuff k such that $G \cap k \subseteq G \cap c$ and c is homotopic to k.

Let H be an edgeless graph and let r be a normal root function in G. We say r is *topologically feasible* if there exists a forest F drawn without crossings in Σ such that for each $v \in V(H)$, the forest F has a component F_v with $r(v) \subseteq V(F_v)$, and $F_v \neq F_w$ for distinct $v, w \in V(H)$.

Theorem 1. *For every surface Σ and integer k, there exists p such that the following holds. Let G be a graph with a normal drawing in a surface Σ with at least two holes, such that at most k vertices of G are drawn in the boundary of Σ, and each cuff contains at least one vertex of G. Let H be an edgeless graph and let r be a normal root function assigning to each vertex of H a non-empty set. If r is topologically feasible and the drawing of G is p-generic, then H is a minor of G rooted in r.*

Our aim now is to get a stronger result in terms of respectful tangles, and to apply it to obtain a polynomial-time algorithm. First, we need a technical result about grid-like substructures in these graphs.

1 Sleeves

Let G be a graph with a 2-cell drawing in a surface Σ and let $\mathcal{T}$ be a respectful tangle in G. Let X be a set of t vertices of G, all incident with the same face f. Let $\mathcal{C} = \{C_0, C_1, \ldots, C_{2p}\}$ be a set of vertex-disjoint cycles and $\mathcal{P} = \{P_1, P_2, \ldots, P_{tp}\}$ a set of vertex-disjoint paths in G. We say $(\mathcal{C}, \mathcal{P})$ is a *sleeve around (f, X) of order p* if

- there exists a disk $\Delta \subseteq \Sigma$ containing f, $\mathcal{C}$, and $\mathcal{P}$ such that $d_{\mathcal{T}}(f, a) \leq (t + 14)p + 10$ for every atom a of G contained in Δ,
- for any $i < j$, C_i separates f from C_j,
- every atom a such that $d_{\mathcal{T}}(f, a) \leq tp$ is drawn between f and C_{2p},

- for any i and j, $C_i \cap P_j$ is a connected path, and
- there exist disjoint paths $Q_1, \ldots, Q_t \subset C_p$, each containing the intersection of C_p with p paths of $\mathcal{P}$, and disjoint paths $L_1, \ldots, L_p$ in G, where L_i has one end in X, the other end in Q_i, and is otherwise disjoint from C_p.

We say that Δ is the *locus* of the sleeve, C_p is the *belt* of the sleeve, and $\mathcal{Q} = \{Q_1, \ldots, Q_t\}$ and $\mathcal{L} = \{L_1, \ldots, L_t\}$ form a *seam* of the sleeve. Recall that a set X is *free* in a tangle $\mathcal{T}$ if there is no separation $(A, B) \in \mathcal{T}$ of order less than $|X|$ such that $X \subseteq V(A)$. We refer to the metric derived from $\mathcal{T}$ as the *$\mathcal{T}$-distance*, to distinguish it from the distance in the graph.

Lemma 2. *For all integers $t \le p$ and every surface Σ without boundary, there exists θ_0 such that the following holds. Let G be a graph with a 2-cell drawing in Σ and let $\mathcal{T}$ be a respectful tangle in G of order $\theta \ge \theta_0$. Let X be a set of t vertices of G, all incident with the same face f of G. If X is free, then G contains a sleeve around (f, X) of order p.*

Proof. Recall that for every $l < \theta$, the union U of atoms of the radial drawing $R(G)$ of G at $\mathcal{T}$-distance at most l from the vertex $R(f)$ corresponding to f is simply-connected. We apply this observation for $l = (t+14)p+10$, and we let $\Delta \subseteq U$ be a maximal disk containing f. Then $d_{\mathcal{T}}(f, a) \le (t+14)p+10$ for every atom a of G contained in Δ, and conversely, every atom a such that $d_{\mathcal{T}}(f, a) \le (t+14)p+8$ is contained in Δ.

Consider now any l such that $3 \le l \le (t+14)p+5$, and let Z_l be the set of vertices v of G such that $|d_{\mathcal{T}}(f, v) - l| \le 1$. Then Z_l separates f from the boundary of Δ, and thus there exists a simple closed curve c_l intersecting G only in vertices of Z_l and separating f from the boundary of Δ. Let W_l be the closed walk in G passing through the vertices $c_l \cap G$, and between them following the boundaries of the faces through which c_l passes. Then $|d_{\mathcal{T}}(f, v) - l| \le 3$ for every $v \in V(W_l)$. Moreover, W_l is homotopic to c_l, and thus there exists a cycle S_l with $V(S_l) \subseteq V(W_l)$ separating f from the boundary of Δ. For $i = 0, \ldots, 2p$, let $C'_i = S_{tp+3+7i}$. Note that every atom a such that $d_{\mathcal{T}}(f, a) \le tp$ is drawn between f and C'_{2p}.

We claim there exist tp pairwise disjoint paths $P'_1, \ldots, P'_{tp}$ from C'_0 to C'_{2p}. Indeed, otherwise by Menger's theorem, there would exist a simple G-normal closed curve c separating C'_0 from C'_{2p} with $|c \cap G| < tp$. Let W be the closed walk in $R(G)$ tracing c. If $f \subset \mathrm{ins}_{\mathcal{T}}(W)$, we would have $V(C'_0) \subset \mathrm{ins}_{\mathcal{T}}(W)$, implying $d_{\mathcal{T}}(f, v) < tp$ for $v \in V(C'_0)$, contradicting the choice of C'_0. Otherwise, for any $v \in V(C'_{2p})$ and any atom a of G not in Δ, we have $d_{\mathcal{T}}(v, a) < tp$, and thus $d_{\mathcal{T}}(f, a) < (2t+14)p+6 < \theta$ for every such atom a; this inequality holds also for all atoms intersecting Δ, contradicting the fact that some edge of G is at $\mathcal{T}$-distance θ from f.

We can now apply (a variation of) the loom cleaning procedure from the second lecture to obtain cycles $\mathcal{C} = \{C_0, C_1, \ldots, C_{2p}\}$ and paths $\mathcal{P} = \{P_1, P_2, \ldots, P_{tp}\}$ such that $C_{2p} = C'_{2p}$, $\bigcup \mathcal{C} \cup \bigcup \mathcal{P} \subseteq \bigcup_{i=1}^{2p} C'_i \cup \bigcup_{j=1}^{tp} P'_j$, for any $i < j$, C_i separates f from C_j, and for any i and j, $C_i \cap P_j$ is a connected path.

Therefore, it remains to find a seam for the sleeve. Choose disjoint paths $Q_1, \ldots, Q_t \subset C_p$ each containing the intersection of C_p with p paths of $\mathcal{P}$ arbitrarily, so that $V(C_p) = V(Q_1) \cup \ldots \cup V(Q_t)$. Let G' be obtained from G by contracting each of the paths Q_i to a single vertex q_i, and let $Y = \{q_1, \ldots, q_t\}$. It suffices to prove that G' contains t disjoint paths from X to Y. If not, Menger's theorem implies there exists a simple closed curve c separating X from Y and intersecting G' in less than t vertices. Note that c cannot pass through a vertex in Y, as otherwise it would have to intersect either $C_0, \ldots, C_{p-1}$ or all paths in $\mathcal{P}$. Consequently, c also intersects G in less than t vertices and separates f from the boundary of Δ. Let W be the closed walk in $R(G)$ corresponding to c. Since X is free, we cannot have $X \subset \mathrm{ins}_{\mathcal{T}}(W)$, and thus $C_{2p} \subset \mathrm{ins}_{\mathcal{T}}(W)$. However, that implies $d_{\mathcal{T}}(f, a) < (t + 14)p + 6 + t < \theta$ for every atom a of G, which is a contradiction. □

2 Minors from a respectful tangle

Theorem 3. *For every surface Σ without boundary and integer k, there exists θ_0 such that the following holds. Let G be a graph with a 2-cell drawing in Σ and let $\mathcal{T}$ be a respectful tangle in G of order $\theta \geq \theta_0$. For some $q \leq k$, let $f_1, \ldots, f_q$ be distinct faces of G and let X be a set of k vertices of G, each incident with one of these faces; let X_i denote the set of vertices of X incident with f_i. Let $\Sigma' = \Sigma - (f_1 \cup \ldots \cup f_q)$. Let H be an edgeless graph and let r be a root function assigning to each vertex of H a non-empty subset of X. If r is topologically feasible in Σ', $d_{\mathcal{T}}(f_i, f_j) \geq \theta_0$ for all distinct i and j and X_i is free for $i = 1, \ldots, q$, then H is a minor of G rooted in r.*

Proof. We can assume $X_i \neq \emptyset$ for all i, as otherwise we can ignore the face f_i. We can also assume $q \geq 2$: if $q = 0$, we can choose f_1 arbitrarily and add an incident vertex to X, increasing q to 1; for $q = 1$, we can choose f_2 at $\mathcal{T}$-distance at least θ_0 from f_1 and add a vertex incident with f_1 to X. The assumption that X_1 and X_2 are free is trivially satisfied, since G is connected. We also make all vertice of X roots by modifying r if necessary. It follows that Σ' has at least two cuffs, each incident with a root vertex.

By Lemma 2, for $i = 1, \dots, q$, G contains a sleeve $(\mathcal{C}_i, \mathcal{P}_i)$ around (f_i, X_i) of order p with locus Δ_i and seam $(\mathcal{Q}_i, \mathcal{L}_i)$. Note that for $i \neq j$, we have $\Delta_i \cap \Delta_j = \emptyset$, since $d_{\mathcal{T}}(f_i, f_j) \geq \theta_0$. Let G' be the graph obtained from G by, for $i = 1, \dots, q$,

- deleting everything in the open disk bounded by the belt of $\mathcal{C}_i$ containing f_i, and
- contracting each path of $\mathcal{Q}_i$ to a single vertex; let X'_i denote the resulting set of vertices, and for $x \in X_i$, let x' be the vertex of X'_i to which it is connected by a path in $\mathcal{L}_i$.

We view G' as drawn in a surface Σ'' homeomorphic to Σ', where the i-th cuff intersects G' exactly in X'_i. Let r' be the root function where, for $z \in V(H)$, $r'(z) = \{x' : x \in r(z)\}$. Note that a minor of H in G' rooted in r' can be transformed into a minor of H in G rooted in r, by decontracting the paths in $\mathcal{Q}_1, \dots, \mathcal{Q}_q$ and adding the paths in $\mathcal{L}_1, \dots, \mathcal{L}_q$. Furthermore, since r is topologically feasible in Σ', r' is topologically feasible in Σ''. Therefore, to finish the proof by using Theorem 1, it suffices to argue that the drawing of G' in Σ'' is p-generic.

For any simple G'-normal curve c with ends in distinct cuffs of Σ', let f_i and f_j be the corresponding faces and C_i and C_j the corresponding belts. We have $|c \cap G'| \geq \frac{1}{2}(d_{\mathcal{T}}(f_i, f_j) - d_{\mathcal{T}}(f_i, C_i) - d_{\mathcal{T}}(f_j, C_j)) \geq \frac{1}{2}(\theta_0 - 2(k+14)p - 20) > p$, as required.

Consider now a simple closed non-contractible G'-normal curve c intersecting G' in less than p vertices. Suppose first that c is disjoint from the cuffs of Σ'', and thus c is also G-normal and intersects G in less than p vertices when drawn in Σ. Let W be the corresponding closed walk in $R(G)$. Since c is non-contractible in Σ'' and $d_{\mathcal{T}}(f_i, f_j) > p$ for distinct i and j, there exists unique i such that $f_i \subset \mathrm{ins}_{\mathcal{T}}(W)$. Hence, $d_{\mathcal{T}}(f_i, v) < p$ for $v \in V(G) \cap c$, and thus c is drawn between f_i and the last cycle in $\mathcal{C}_i$. If c intersects the cuff, we obtain the same conclusion since c cannot intersect all cycles in $\mathcal{C}_i$ between the belt and the last one.

If there existed $x' \in X'_i$ not belonging to c, then let $P_1, \dots, P_p$ be the paths of $\mathcal{P}_i$ intersecting the path of $\mathcal{Q}_i$ that was contracted to x'. Then c must intersect all of $P_1, \dots, P_p$, contradicting the assumption $|G \cap c| < p$. We conclude that $X'_i \subseteq G \cap c$, confirming that the drawing of G' in Σ'' is p-generic. □

Let us now give a simple application.

Corollary 4. *For every surface Σ without boundary and a graph H drawn in Σ, there exists θ_1 such that the following holds. Let G be a 2-connected* *graph with a 2-cell drawing in Σ and let $\mathcal{T}$ be a respectful tangle in G of order $\theta \geq \theta_1$. Let r be a root function such that $r(x)$ consists of a single vertex v_x for every $x \in V(H)$. If $d_{\mathcal{T}}(v_x, v_y) \geq \theta_1$ for every distinct $x, y \in V(H)$, then G contains H as a minor rooted in r.*

Proof. Let $k = |V(H)|$ and $m = |E(H)|$. There exists edges e and e' of G such that $d_{\mathcal{T}}(e, e') \geq \theta_1$, and thus on a path from e to e' in G, we can find edges $e_1, \ldots, e_{k+m}$ such that $d_{\mathcal{T}}(e_i, e_j) \geq \frac{\theta_1}{4(k+m)}$ for distinct i and j. Each vertex v_x is at $\mathcal{T}$-distance less than $\frac{\theta_1}{8(k+m)}$ from at most one of these edges, and thus we can assume that for $i = 1, \ldots, m$, the $\mathcal{T}$-distance between e_i and v_x is at least $\frac{\theta_1}{8(k+m)}$ for every $x \in V(H)$. Assign to each edge $h = xy \in E(H)$ one of these edges and denote its ends h_x and h_y. Note that $\{h_x, h_y\}$ is free, since G is 2-connected. Let H' be the edgeless graph with $V(H') = V(H)$, and let r' be the root function such that for each $x \in V(H')$, $r'(x)$ consists of v_x and the vertices h_x for all edges h of H incident with x. Applying Theorem 3, we obtain a minor of H' in G rooted in r'. In combination with the edges $e_1, \ldots, e_m$, this gives a minor of H in G rooted in r. □

3 Algorithm

Suppose we are given a graph G drawn normally in a surface Σ with boundary and an edgeless graph H with a normal root function r, and we want to decide whether H is a minor of G rooted in r. We will construct the algorithm inductively according to the *complexity* of the surface—the triple (g, h, k), where g is the genus of the surface, h is the number of holes, and k is the number of root vertices, sorted lexicographically.

The basic operation we use is *cutting*: Suppose for example that there exists a non-contractible separating G-normal curve c such that $|G \cap c| \leq k'$, for some k' depending only on (g, h, k). There are only finitely many ways how a minor of H can intersect $G \cap c$, and for each of them, we obtain a problem of the form: do prescribed rooted minors exist in both graphs into which G is cut along c? Both of these subproblems can be solved recursively, since each of the resulting surfaces has complexity at most $(g-2, h+1, k+k') \prec (g, h, k)$.

We aim to keep simplifying the instance by cutting until Theorem 3 can be applied, or until we reduce to one of the cases we already dealt with in the previous lecture or in the homework assignment (disk, cylinder). Let Σ' denote the surface obtained from Σ by patching each cuff, let $f_1, \ldots, f_h$ be the faces corresponding to these patches, and for $i = 1, \ldots, h$, let X_i be the set of roots incident with f_i. Let us go over each of the assumptions of

Theorem 3 and present a reduction in case it is not satisfied:

- G does not contain a respectful tangle $\mathcal{T}$ of order θ_0. If Σ' is the sphere, this implies G has treewidth at most $\frac{3}{2}\theta_0$, and we can apply an algorithm for graphs with bounded treewidth (the fact that H is a rooted minor of G can be expressed in monadic second-order logic). If Σ' is not the sphere, this implies G drawn in Σ' has representativity less than θ_0. Cutting along the corresponding curve reduces the problem to subproblems of complexity at most $(g-1, h+2, k+2\theta_0) \prec (g, h, k)$.

- r is not topologically feasible in Σ: Then H cannot appear in G as a rooted minor.

- $d_{\mathcal{T}}(f_i, f_j) < \theta_0$ for some distinct i and j. Let W be a tie in $R(G)$ certifying this. If W is a path from f_i to f_j, then cutting along W reduces the problem to subproblems of complexity $(g, h-1, k+2\theta_0) \prec (g, h, k)$. If W is a lollipop or a dumbbell, then cutting along W reduces the problem to subproblems of complexity $(g, h-1, k+2\theta_0) \prec (g, h, k)$ and to ones in a cylinder.

- If X_i is not free, then there exists a cycle W in $R(G)$ intersecting G in less than $|X_i|$ vertices and such that $X_i \subset \text{ins}_{\mathcal{T}}(W)$. Cutting along W educes the problem to subproblems of complexity at most $(g, h, k-1) \prec (g, h, k)$ and to ones in a cylinder.