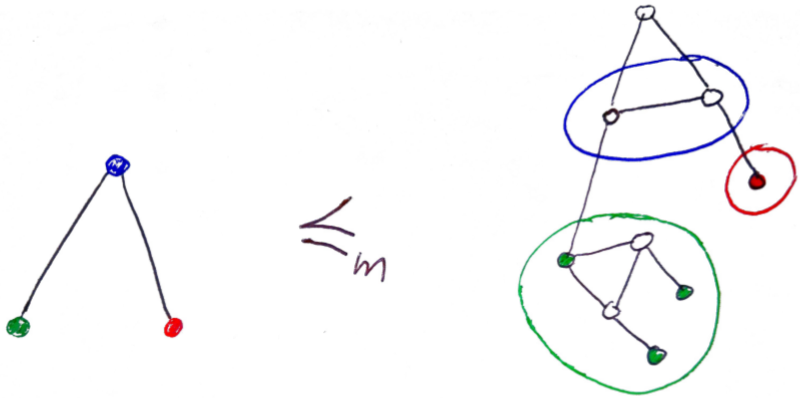
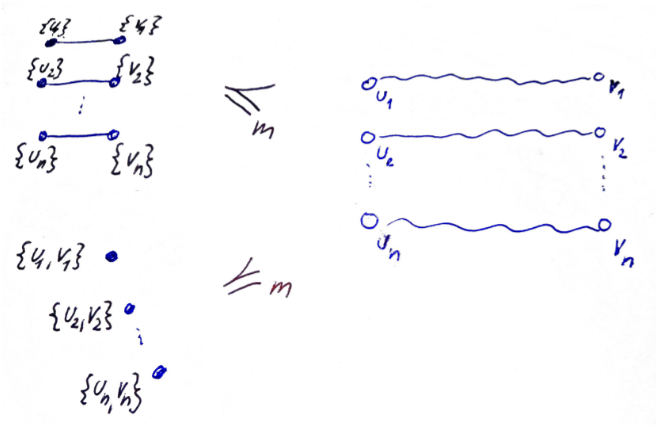
Definition

Model μ of a minor of H in G is a function s.t.

- $\mu(v_1), \ldots, \mu(v_k)$ (where $V(H) = \{v_1, \ldots, v_k\}$ are vertex-disjoint connected subgraphs of G, and
- for $e = uv \in E(H)$, $\mu(e)$ is an edge of G with one end in $\mu(u)$ and the other in $\mu(v)$.

For $r : V(H) \to 2^{V(G)}$ such that $r(u) \cap r(v) = \emptyset$ for distinct u, v, the model is rooted in r if $r(v) \subseteq V(\mu(v))$ for each $v \in v(H)$.


≤m



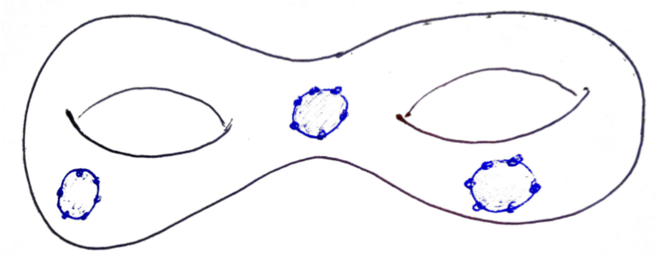
- $V(H) = \{s_1, t_1, \ldots, s_n, t_n\}$, $E(H) = \{s_1 t_1, \ldots, s_n t_n\}$, $r(s_i) = \{u_i\}$, $r(t_i) = \{v_i\}$.
- $V(H) = \{p_1, p_n\}$, $E(H) = \emptyset$, $r(p_i) = \{u_i, v_i\}$.

Goal

Theorem

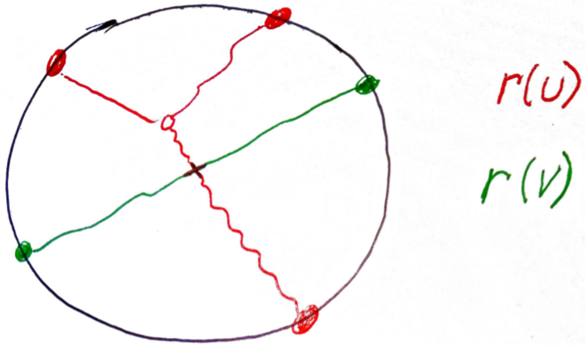
For every H with vertices $v_1, \ldots, v_k$ drawn in a surface Σ, there exists θ such that the following holds. If G is drawn in Σ has a respectful tangle $\mathcal{T}$ of order θ, $r(v_i) = \{u_i\}$ for $i = 1, \ldots, k$, and $d_{\mathcal{T}}(u_i, u_j) = \theta$ for $i \neq j$, then G has a minor of H rooted in r.

For a surface Σ with holes, the components of the boundary are cuffs.



Drawing in Σ is normal if it intersects the boundary only in vertices. Root assignment r is normal if $r(v) \subset$ boundary for each v.

G drawn normally in the disk, $v_1, v_2, \ldots, v_m$ vertices in the cuff. Root assignment r is topologically infeasible if for some $i_1 < i_2 < i_3 < i_4$ and $u \neq v$, $v_{i_1}, v_{i_3} \in r(u)$ and $v_{i_2}, v_{i_4} \in r(v)$.

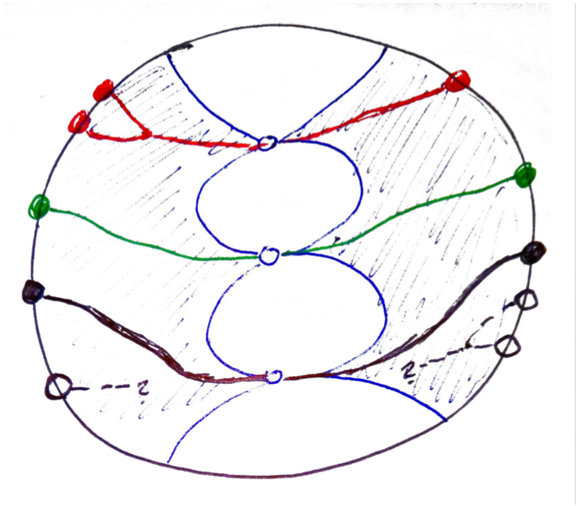


Topologically feasible otherwise.

A *G-slice*: simple G-normal curve c intersecting the cuff exactly in its ends, splits the disk into Δ_1 and Δ_2.

$r/c = \{v : r(v) \cap \Delta_1 \neq \emptyset \neq r(v) \cap \Delta_2$.

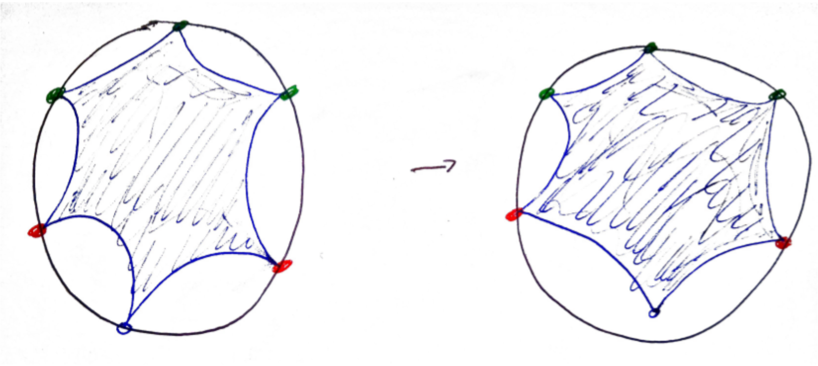
Connectivity-wise feasible: $|G \cap c| \geq |r/c|$ for every G-slice c.



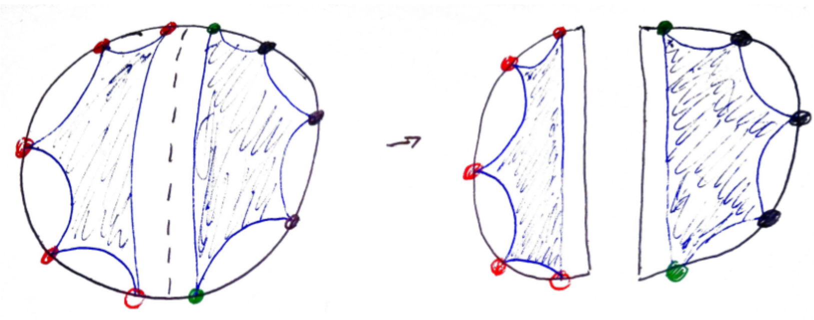
Theorem

G normally drawn in a disk, r normal root function. Topologically and connectivity-wise feasible $\Rightarrow$ edgeless minor rooted in r.

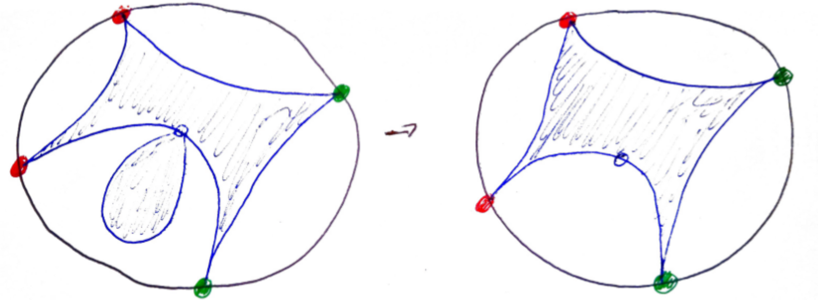
We can assume $G \cap \text{cuff} = \text{roots}$.



G-slice disjoint from *G* splitting *G* into two parts:

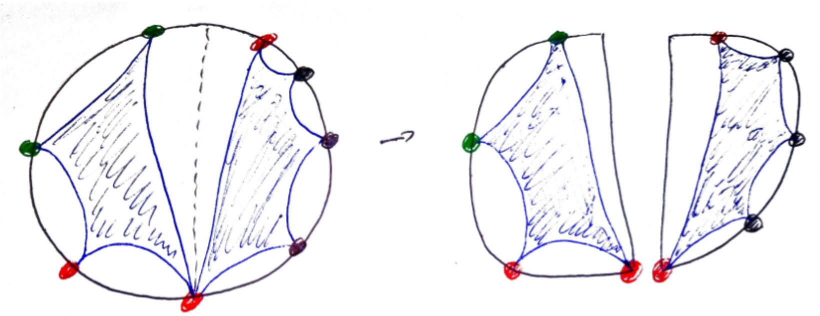


Simple closed curve intersecting G in just one vertex, interior contains a part of G:

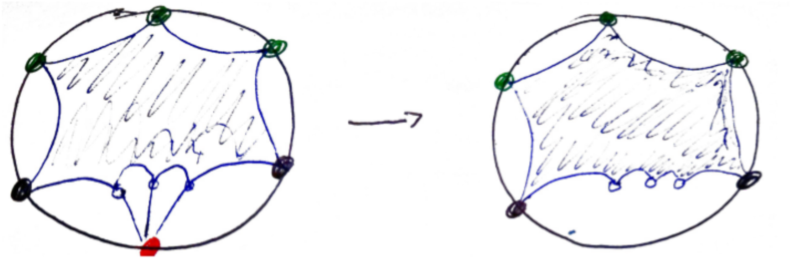


We can assume: faces intersecting cuffs are bounded by paths.

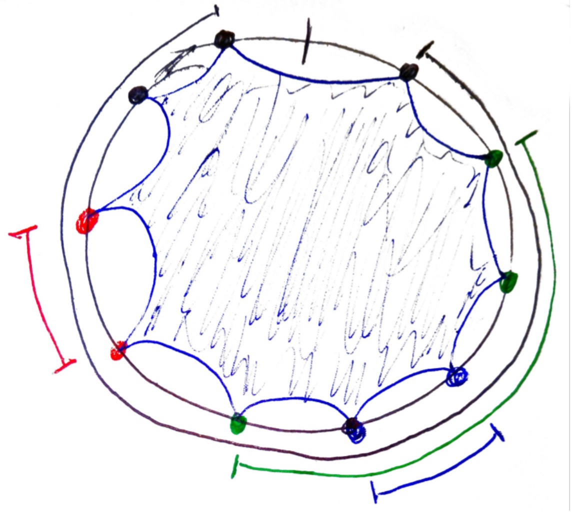
G-slice intersecting *G* in a root, splitting *G* into two parts:



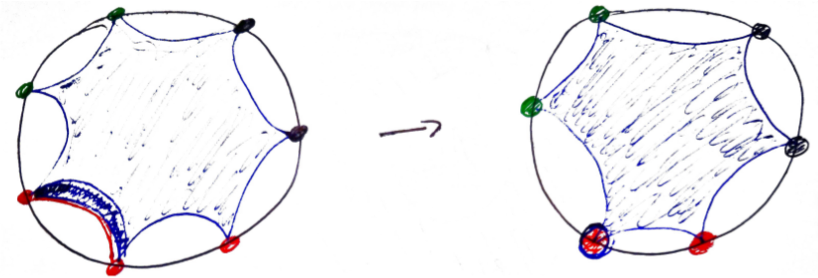
$|r(y)| = 1$:



Select y spanning minimal arc:

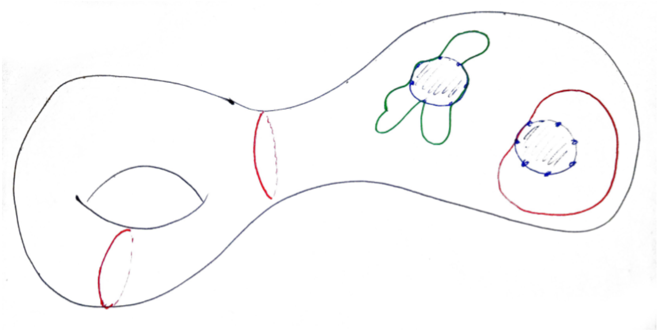


Contract path between consecutive vertices of $r(y)$:

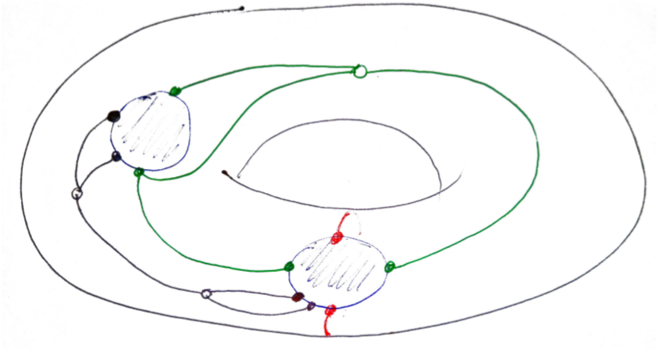


In a surface Σ, a normal drawing of G is p-generic if

- curves between distinct cuffs intersect G at least p times
- simple closed G-normal non-contractible curve c intersects G in $< p$ points $\Rightarrow$ for a cuff k homotopic to k, $G \cap k \subseteq G \cap c$.



A normal root assignment r is topologically feasible if there exists a forest with components $F_v : v \in \mathrm{dom}(r)$ drawn in Σ such that $r(v) \subseteq V(F_v)$.



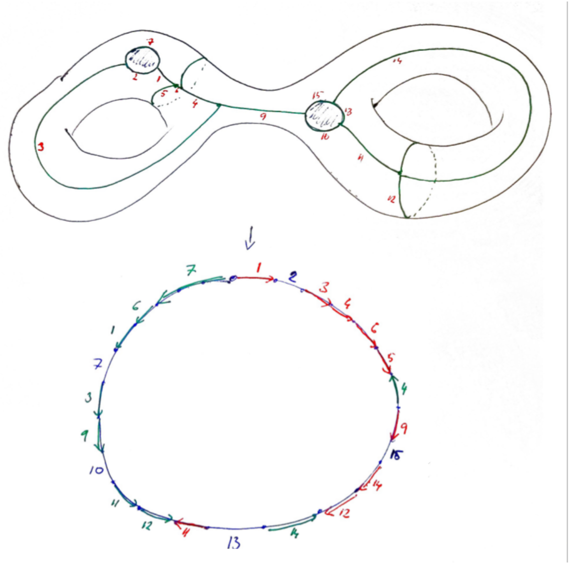
Theorem

$(\forall \Sigma, k)(\exists p)$: *Let G be a graph with a normal drawing in a surface* Σ *with at least two holes, at most* k *vertices in the boundary of* Σ*, each cuff contains at least one vertex. Normal root assignment* r *is topologically feasible and the drawing of G is p-generic* $\Rightarrow$ *edgeless minor rooted in* r*.*

g genus, h number of holes of Σ, $k \ll s \ll p$

G-net *N* drawn in Σ so that

- $N \cap G = V(N) \cap V(G)$,
- each cuff traces a cycle in *N*, and
- *N* has exactly one face, homeomorphic to an open disk.



N with $|G \cap N|$ minimum, subject to that with $|V(N)|$ minimum.

- N connected, minimum degree at least two.
- One face $\Rightarrow$ only non-contractible cycles.
- At least two cuffs: not a cycle.

N': suppress vertices of degree two in N.

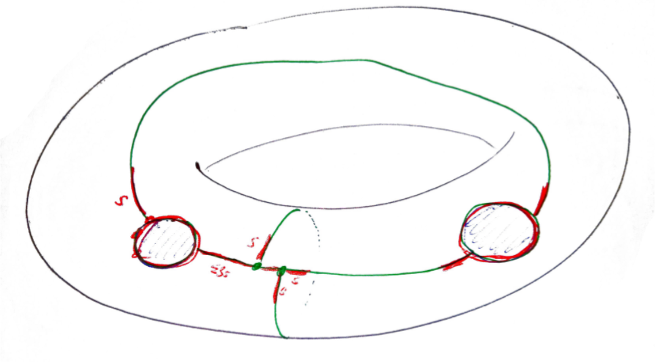
- Minimum degree at least three: $|E(N')| \geq \frac{3}{2}|V(N')|$.
- One face, h holes: $|E(N')| = |V(N')| + (h+1) + g - 2$.
- $|V(N')| \leq 2(g+h-1)$, $|E(N')| \leq 3(g+h-1)$.

$X =$ vertices of N of degree at least three or contained in cuffs:

$$|X| \leq 2(g+h) + k$$

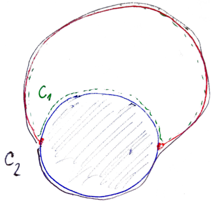
$S =$ the subgraph of N consisting of

- paths of length at most s starting in X, and
- paths of length at most $3s$ between the vertices of X.



$$|V(S)| \leq 9(g+h)s \ll p$$

Drawing of G is p-generic, all cycles in N are non-contractible:

- No path in S internally disjoint from the cuffs has both ends in cuffs. 

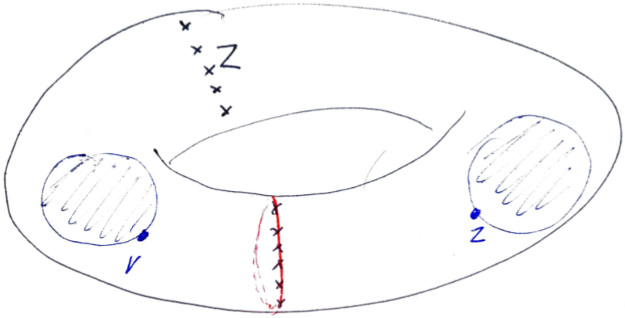
- Every cycle in S bounds a cuff.

Each component of S is either

- a tree, or
- unicyclic with the cycle tracing a cuff.

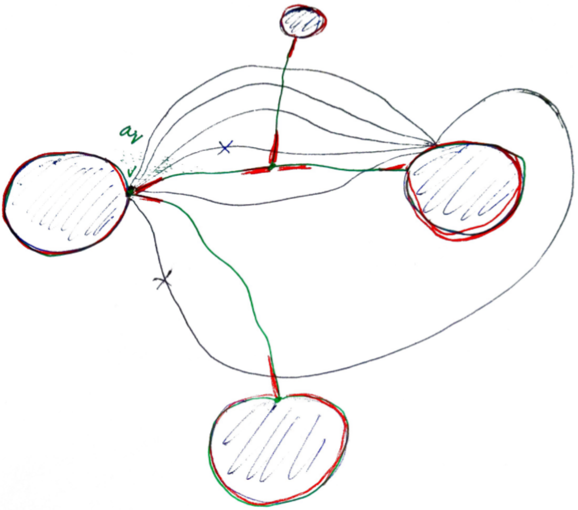
For each v in a cuff, there exist p disjoint paths from v to a vertex z in another cuff.

- Otherwise, separated by a set Z of less than p vertices.
- Non-contractible curve through Z contradicting p-genericity of G.



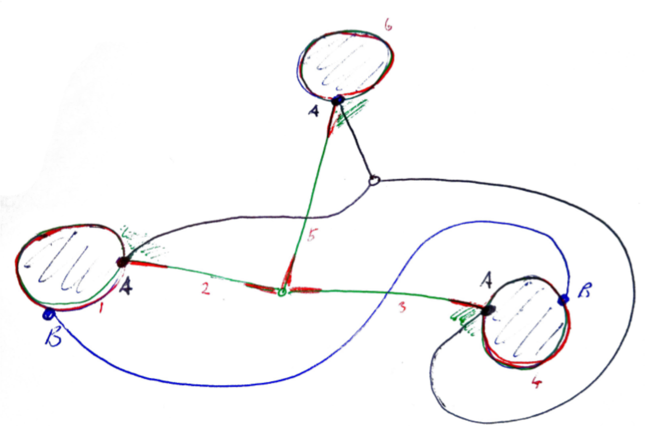
At least $\frac{p-|V(S)|}{|V(S)|} \geq s$ of the paths are

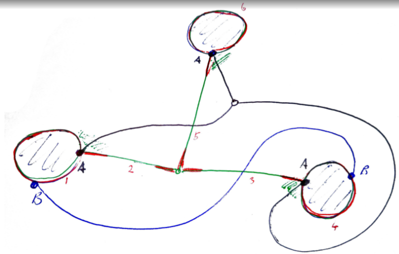
- internally disjoint from S, and
- leaving v through the same angle a_v of N.

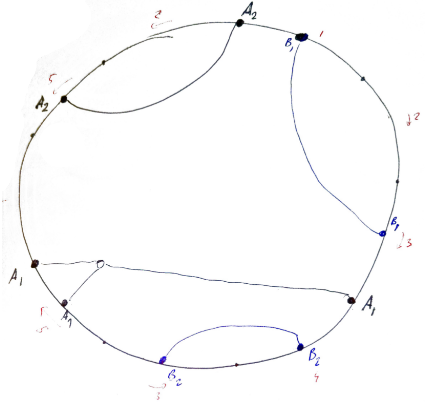


The forest F certifying topological feasibility of r can be shifted so that

- F is disjoint from S except for the cuffs,
- F intersects N in at most $\gamma_{\Sigma,k} \ll s$ vertices, and
- for v in a cuff, all edges of F leave through the angle a_v.




6
A
5
A
2
3
A
B
B
1
4


2
A2
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A1
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3

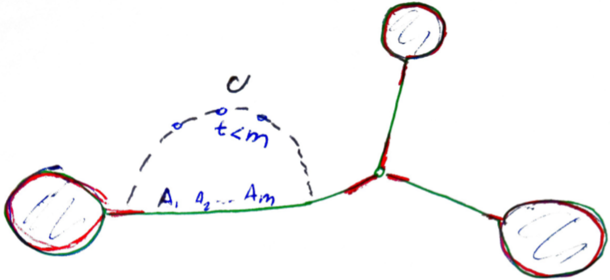
- Cut Σ along N, obtaining G' in a disk.
- r': According to components into which F is cut.
- Apply the disk theorem.
- Topological feasibility from the choice of r'.
- We need to verify connectivity-wise feasibility.

For contradiction: G'-slice c, intersecting G' in $t < |r'/c|$ vertices.

- $|r'/c| \leq 2\gamma_{\Sigma,k} \ll s$.
- $N \cup c$ has two faces, cycle C separating them.

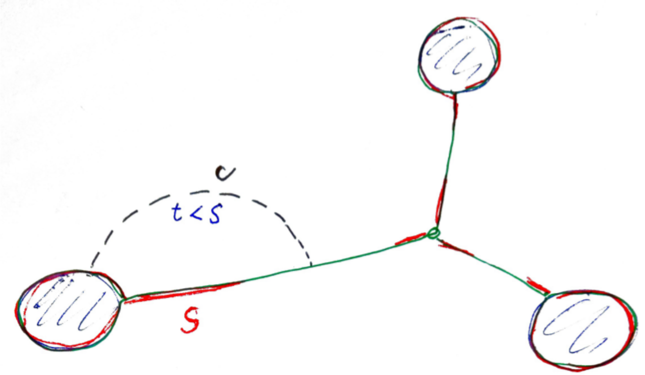
Case 1: $C \cap X = \emptyset$ $\Rightarrow$ $C - c$ is a path of at least $|r'/c|$ vertices of degree two in N.

Replacing $C - c$ by c in N gives a net contradicting the minimality of N.



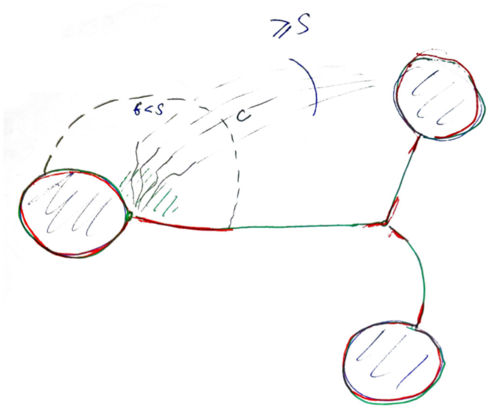
Case 2: $C \cap X \neq \emptyset$, $C \not\subseteq S \cup c$ $\Rightarrow$ $C - c$ contains a path R of $s \gg |r'/c|$ vertices of degree two.

Replacing R by c in N gives a net contradicting the minimality of N.



Case 3: $C \cap X \neq \emptyset$, $C \subseteq S \cup c$

- $r'/c \neq \emptyset \Rightarrow C$ contains a vertex v in a cuff
- The angle a_v in the disk bounded by C.
- More than s paths internally disjoint from S through a_v.
- Contradiction with $t < |r'/c| \leq s$.



We have:

Theorem

$(\forall\Sigma, k)(\exists p)$: *Let G be a graph with a normal drawing in a surface Σ with at least two holes, at most k vertices in the boundary of Σ, each cuff contains at least one vertex. Normal root assignment r is topologically feasible and the drawing of G is p-generic $\Rightarrow$ edgeless minor rooted in r.*

Want:

- Get rid of "at least two holes", "each cuff contains a vertex".
- Weaken the p-generic assumption: For a curve c surrounding a cuff k, only require $|G \cap c| \geq |G \cap k|$.