Tangles

Definition

Tangle $\mathcal{T}$ of order $\theta =$ set of separations of G of order less than θ s.t.

(T1) $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$ for every separation (A, B) of order less than θ.

(T2) $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T} \Rightarrow A_1 \cup A_2 \cup A_3 \neq G$.

(T3) $(A, B) \in \mathcal{T} \Rightarrow V(A) \neq V(G)$.

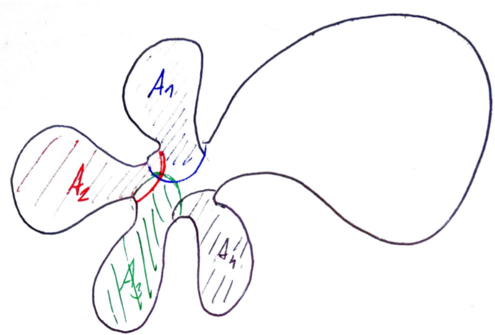
Definition

Pre-tangle: Only satisfies (T1) and (T2).

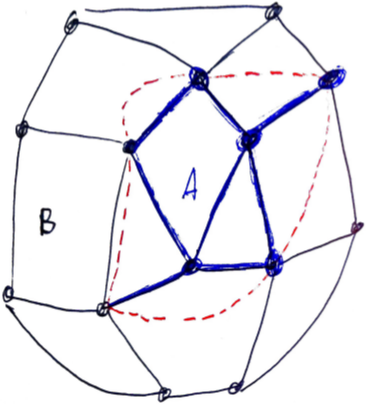
Lemma

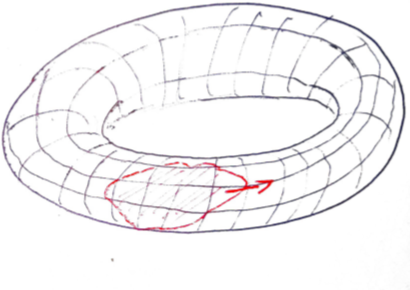
$\mathcal{T}$ pre-tangle of order θ. Suppose $(A_1, B_1), \ldots, (A_m, B_m) \in \mathcal{T}$ and $\left|\bigcup_{i=1}^{m} V(A_i \cap B_i)\right| < \theta$. Then

$$\left(\bigcup_{i=1}^{m} A_i, \bigcap_{i=1}^{m} B_i\right) \in \mathcal{T}.$$

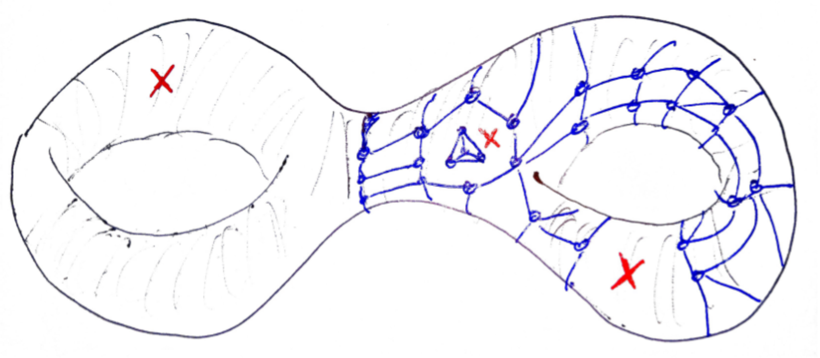


Tangle(?) in an embedded graph

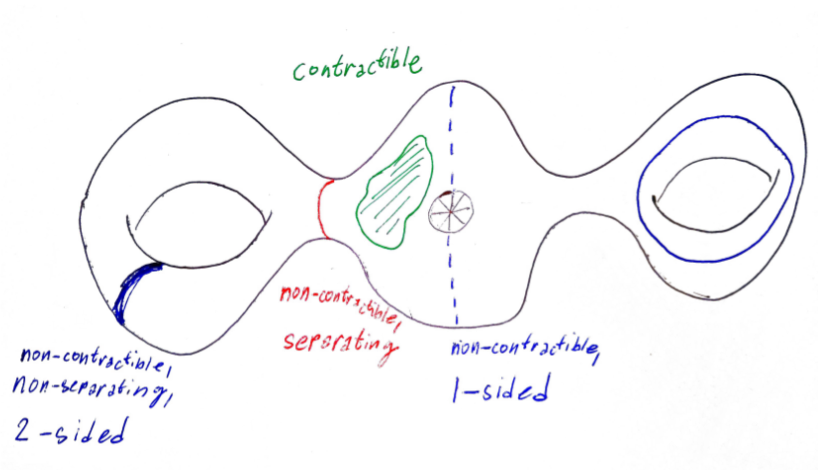




Drawing is 2-cell if all faces are open disks.

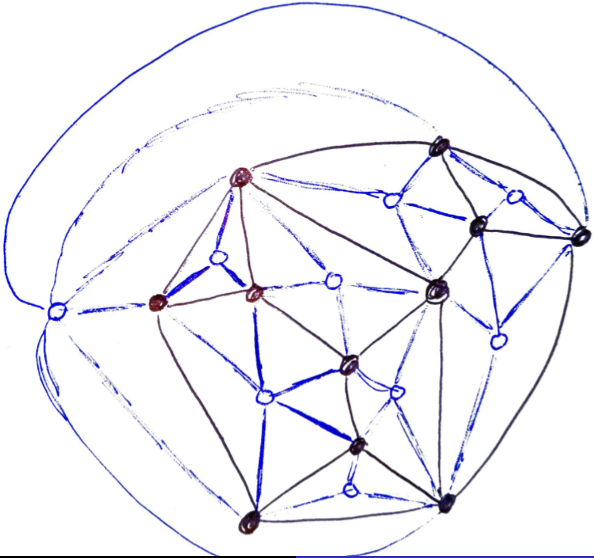


Closed curves



Representativity = minimum number of intersections of G with a non-contractible closed curve.

A curve is *G-normal* if it intersects G only in vertices.
Radial graph: $V(R(G)) = V(G) \cup F(G)$, $E(R(G)) =$ incidence between vertices and faces.



- vertices of $G \leftrightarrow$ one part of $V(R(G))$
- faces of $G \leftrightarrow$ the other part of $V(R(G))$
- edges of $G \leftrightarrow$ the faces of $R(G)$

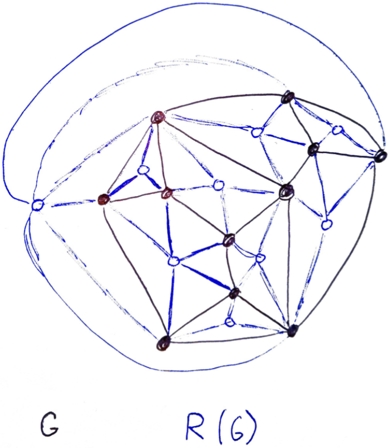
atoms $A(G)$ of G. $R(a) =$ the corresponding object in $R(G)$.

Observation

G-normal curves correspond to walks in $R(G)$.

Observation

$R(G) = R(G^\star)$.



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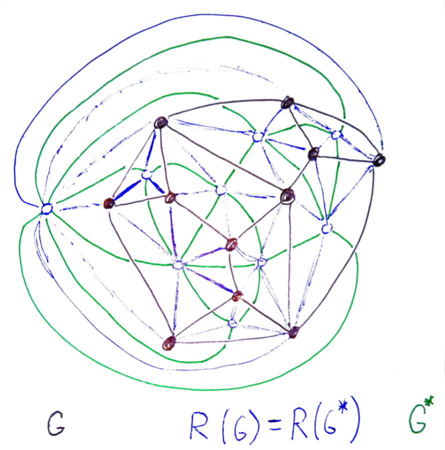


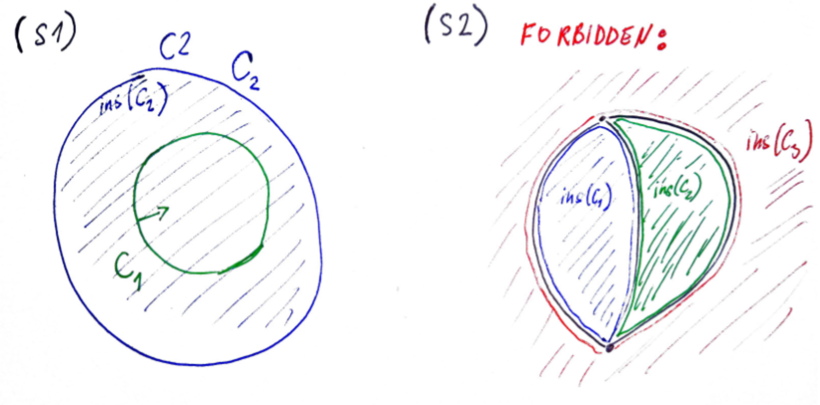
H: 2-cell drawing in Σ.

Definition

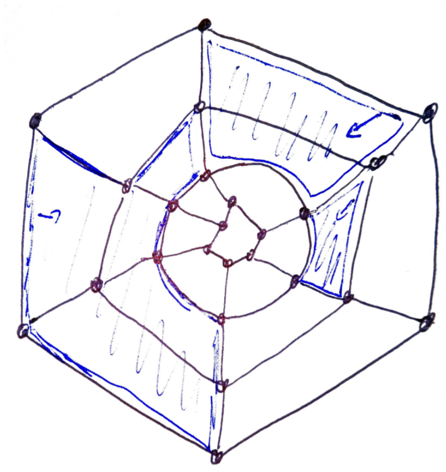
A slope ins of order θ assigns to each cycle $C \subseteq H$ of length less than 2θ a closed disk ins$(C) \subseteq \Sigma$ bounded by C, s.t.

(S1) $\ell(C_1), \ell(C_2) < 2\theta$, $C_1 \subseteq \text{ins}(C_2) \Rightarrow \text{ins}(C_1) \subseteq \text{ins}(C_2)$

(S2) $F \subseteq H$ a theta graph, all cycles in F have length less than $2\theta \Rightarrow$ for some $C \subseteq F$, every cycle $C' \subseteq F$ satisfies $\text{ins}(C') \subseteq \text{ins}(C)$.


(S1)
C2
C2
ins(C2)
C1
(S2)
FORBIDDEN:
ins(C3)
ins(C1)
ins(C2)

- Σ not the sphere: Slope exists iff every non-contractible cycle has length at least 2θ; ins unique.
- Σ is the sphere: "Degenerate" slopes.



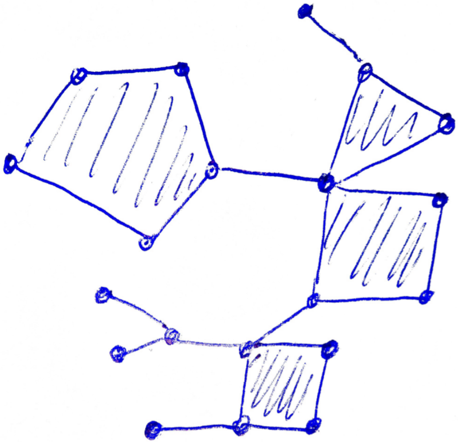
$F \subseteq H$ is confined if all cycles in F have length less than 2θ.

$$\mathrm{ins}(F) = F \cup \bigcup_{C \subseteq F} \mathrm{ins}(C).$$

(S2): F confined $\Rightarrow \mathrm{ins}(F) = \mathrm{ins}(C)$ for some cycle C in F.

Lemma

There exists a cactus $F' \subseteq F$ such that $ins(F) = ins(F')$, and for any distinct 2-connected blocks B_1 and B_2 of F', $ins(B_1)$ and $ins(B_2)$ intersect in at most one vertex. For some face f of F, $ins(F) = \Sigma \setminus f$.



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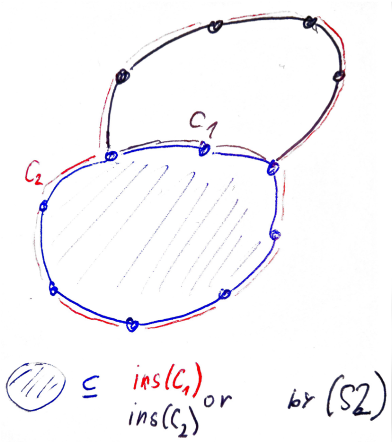
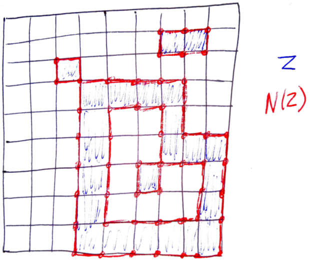


Z a set of faces of H. $N(Z)$: Vertices and edges incident with both Z and $\overline{Z}$.



H bipartite, X one of parts.

Definition

A set Z of faces is X-small if $|V(N(Z)) \cap X| < \theta$ and $Z \subset \text{ins}(N(Z))$.

Lemma

Z_1, Z_2, Z_3 X-small $\Rightarrow Z_1 \cup Z_2 \cup Z_3 \neq$ all faces of H.

Proof.

Complicated. Basic case:

- F theta-subgraph, Z_i faces of H inside one of faces of F.
- $Z_1 \cup Z_2 \cup Z_3 =$ all faces of H.
- $N(Z_i) =$ cycle bounding the i-th face of F.
- By (S2), one of Z_1, Z_2, Z_3 is not small.

□

G with 2-cell drawing in Σ. For a closed disk Δ whose boundary is G-normal,

$$(A_\Delta, B_\Delta) = (G \cap \Delta, G \cap \overline{\Sigma \setminus \Delta}).$$

$\mathcal{T}$: a pre-tangle or tangle of order θ in G.

Definition

$\mathcal{T}$ is respectful if every cycle $C \subseteq R(G)$ of length less than 2θ bounds a disk $\Delta \subseteq \Sigma$ such that $(A_\Delta, B_\Delta) \in \mathcal{T}$.

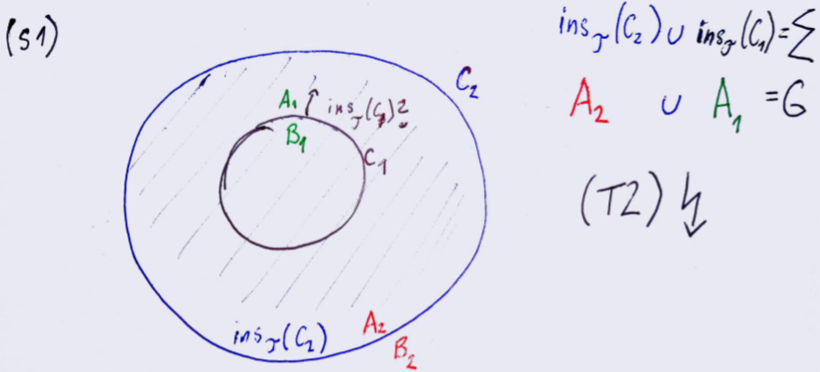
We define $\text{ins}_{\mathcal{T}}(C) = \Delta$.

- $\Sigma \neq$ the sphere: Implies representativity $\geq \theta$, Δ unique.
- $\Sigma =$ the sphere: Always true.

Lemma

$\mathcal{T}$ respectful pre-tangle of order θ in $G \Rightarrow ins_{\mathcal{T}}$ is a slope of order θ in $R(G)$.

Proof.



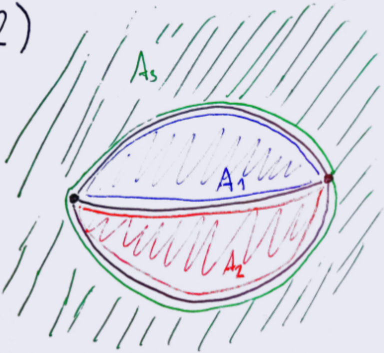
□

Lemma

$\mathcal{T}$ respectful pre-tangle of order θ in $G \Rightarrow ins_{\mathcal{T}}$ is a slope of order θ in $R(G)$.

Proof.

(S2)



$A_1 \cup A_2 \cup A_3 = G$

(T2) ↯

□

From a slope to a pre-tangle

For $A \subseteq G$, let Z_A be the faces of $R(G)$ corresponding to the edges of A.

ins: a slope of order θ in $R(G)$

Definition

$\mathcal{T}_{\text{ins}}$ = the set of separations (A, B) of order less than θ such that Z_A is $V(G)$-small in $R(G)$.

Note:

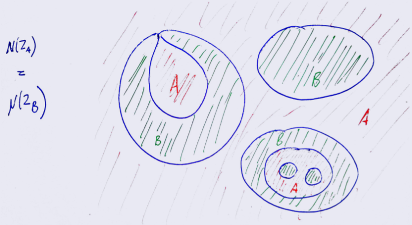
$$\begin{aligned} V(N(Z_A)) \cap V(G) &= \text{vertices incident with both } E(A) \text{ and } E(B) \\ &\subseteq V(A \cap B). \end{aligned}$$

Lemma

ins is a slope of order θ in $R(G) \Rightarrow \mathcal{T}_{ins}$ is a respectful pre-tangle of order θ in G.

Proof.

(T1) $\text{ins}(N(Z_A))$ is a complement of a face of $N(Z_A)$, $N(Z_A) = N(Z_B) \Rightarrow Z_A$ or Z_B is $V(G)$-small.



(T2) Union of three $V(G)$-small sets does not contain all faces.

Respectfulness: Z_1, Z_2 partition of $F(R(G))$ with $N(Z_1) = C = N(Z_2)$, Z_1 or Z_2 is small. □

$1:1$ correspondence

Lemma

$\mathcal{T}$ respectful pre-tangle of order θ in G:

$$\mathcal{T}_{ins_{\mathcal{T}}} = \mathcal{T}\,.$$

Lemma

ins slope of order θ in $R(G)$:

$$ins_{\mathcal{T}_{ins}} = ins\,.$$

A slope in $R(G)$ is degenerate if for some face f bounded by a 4-cycle C,

$$\text{ins}(C) \neq \text{the closure of } f.$$

Lemma

For $\theta \geq 3$, $\mathcal{T}_{ins}$ is a tangle if and only if ins is non-degenerate.

Proof.

$\Rightarrow$ f of $R(G)$ corresponds to $e \in E(G)$.

By (T3) and (T1), $(e, G - e) \in \mathcal{T}_{\text{ins}}$, so $\text{ins}(C) = \text{the closure of } f$.

□

A slope in $R(G)$ is degenerate if for some face f bounded by a 4-cycle C,

$$\mathrm{ins}(C) \neq \text{the closure of } f.$$

Lemma

For $\theta \geq 3$, $\mathcal{T}_{ins}$ is a tangle if and only if ins is non-degenerate.

Proof.

$\Leftarrow$ By the assumption, $(e, G-e) \in \mathcal{T}_{\mathrm{ins}}$ for every $e \in E(G)$. If $(A, B) \in \mathcal{T}_{\mathrm{ins}}$ and $V(A) = V(G)$, then

$$(G, V(B)) = \left(A \cup \bigcup_{e \in E(B)} e, B \cap \bigcap_{e \in E(B)} G - e \right) \in \mathcal{T}_{\mathrm{ins}},$$

contradicting (T2).

□

Theorem

G 2-cell drawing in $\Sigma \neq$ the sphere.

G contains a respectful tangle of order $\theta \geq 3$ iff the representativity is at least θ. This respectful tangle is unique.

Proof.

The unique slope is non-degenerate. □

Theorem

If G is a plane graph, then G and $G^\star$ have the same branchwidth, and thus their treewidths differs by a factor of at most $3/2$.

Proof.

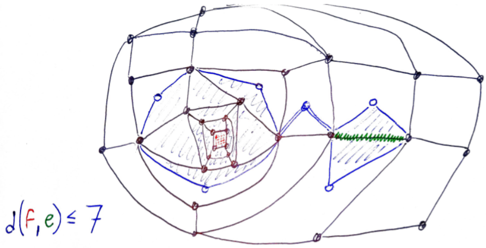
Tangles in G and $G^\star$ correspond to non-degenerate slopes in $R(G) = R(G^\star)$, branchwidth $=$ maximum order of a tangle. □

For a closed walk W in $R(G)$: $G[W]$ = the subgraph on vertices and edges of W, $\text{ins}(W) = \text{ins}(G[W])$.

Definition

For $a, b \in A(G)$,

- $d(a, b) = 0$ if $a = b$,
- $d(a, b) = \ell/2$ if $\exists$ a closed walk W in $R(G)$, $\ell(W) < 2\theta$, such that $R(a), R(b) \, \text{ins}_{\mathcal{T}}(W)$, and ℓ is the length of the shortest such walk,
- $d(a, b) = \theta$ otherwise.



Homework assignment:

- d is a metric
- It suffices to take into account limited types of walks (ties).
- For each $a \in A(G)$ and $k < \theta$, the set

$$\bigcup_{b \in A(G), d(a,b) \leq k} R(b)$$

is “almost a disk”.
- For each $a \in A(G)$, there exists $e \in E(G)$ s.t. $d(a, b) = \theta$.