Flows and linkages

Observation

Edge congestion a, maximum degree $\Delta \Rightarrow$ *vertex congestion* $\leq \Delta a + 1$.

Observation

Flow of size s and vertex congestion c $\Rightarrow$ *flow of size* s/c *and vertex congestion* $1 \Rightarrow (A - B)$*-linkage of size* $\geq s/c$.

Definition

Set W is a-well-linked/node-well-linked if for all $A, B \subset W$ disjoint, of the same size, there exists a flow from A to B of size $|A|$ and edge congestion $\leq a$ / a total $(A - B)$-linkage.

Observation

Either W is a-well-linked, or there exists $X \subseteq V(G)$ such that number of edges leaving $X < a \min(|W \cap X|, |W \setminus X|)$.

Definition

Disjoint sets A and B are node-linked if for all $W \subseteq A$ and $Z \subset B$ of the same size, there exists a total $(W - Z)$-linkage.

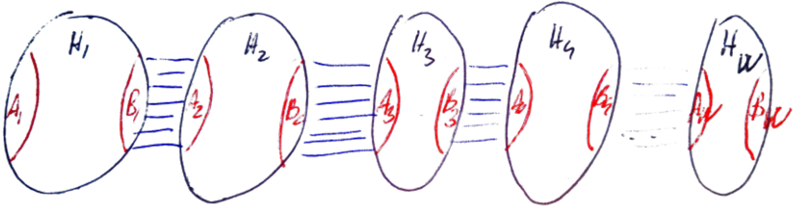
Definition

(G, A, B) a brick of height h if A, B disjoint and $|A| = |B| = h$.
Node-linked if

- Both A and B are node-well-linked.
- A and B are node-linked.

a-well-linked if $A \cup B$ is a-well-linked.

Path-of-sets system



Lemma

a-well-linked path-of-sets system of height at least $16(\Delta a + 1)^2 h \Rightarrow$ node-linked one of height h.

Theorem

Node-linked path-of-sets system of width $2n^2$ and height $2n(6n + 9)$ implies a minor of W_n.

Homework:

Theorem

If G has treewidth $\Omega(t^4 \sqrt{\log t})$, then G contains a subgraph of maximum degree at most four and treewidth at least t.

Theorem (Chekuri and Chuzhoy)

If G has treewidth $\Omega(t \operatorname{polylog} t)$, then G contains a subgraph H of maximum degree at most three and treewidth at least t. Moreover, H contains a node-well-linked set of size t, and all vertices of this set have degree 1 in H.

- Advantage: edge-disjoint paths $\sim$ vertex-disjoint paths.
- Gives a node-linked path-of-sets system of width 1 and height $t/2$.

The doubling theorem

Theorem

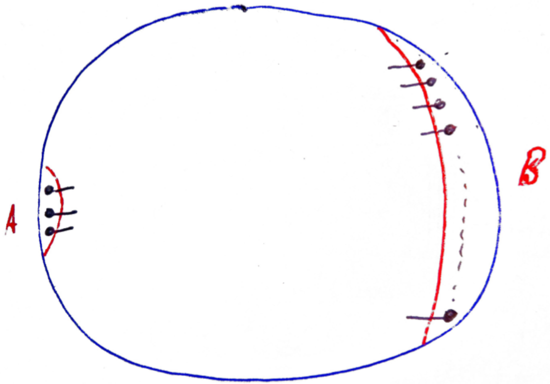
Node-linked path-of-sets system of width w and height $h \Rightarrow$ 64-well-linked path-of-sets system of maximum degree three, width $2w$ and height $h/2^9$.

- Iterate doubling and making the system node-linked.
- After $\Theta(\log n)$ iterations: width $2n^2$, height $h/n^c \geq 2n(6n+9)$

Definition

A good semi-brick of height h is (G, A, B), where A, B are disjoint,

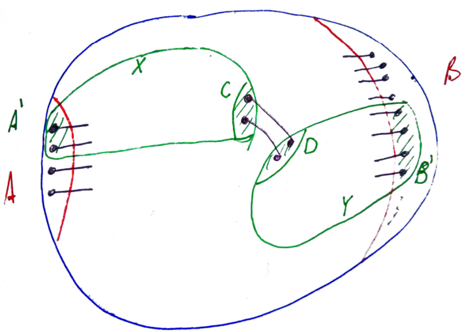
- vertices in A and B have degree 1,
- $|A| = h/64$ and $|B| = h$,
- A and B are node-linked and B is node-well-linked in G.



Definition

A splintering of a semi-brick (G, A, B) of height h:

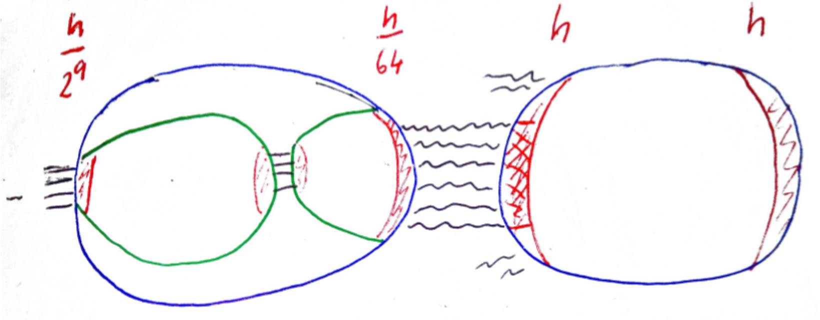
- X and Y disjoint induced subgraphs of G
- $A' \subset A \cap V(X)$ of size $h/2^9$, $B' \subset B \cap V(Y)$ of size $h/64$
- $C \subset V(X) \setminus A'$ and $D \subset V(Y) \setminus B'$ of size $h/2^9$
- perfect matching between C and D in G
- $A' \cup C$ 64-well-linked in X, $D \cup B'$ $(64, \frac{h}{512})$-well-linked in Y.



Theorem

Every good semi-brick has a splintering.

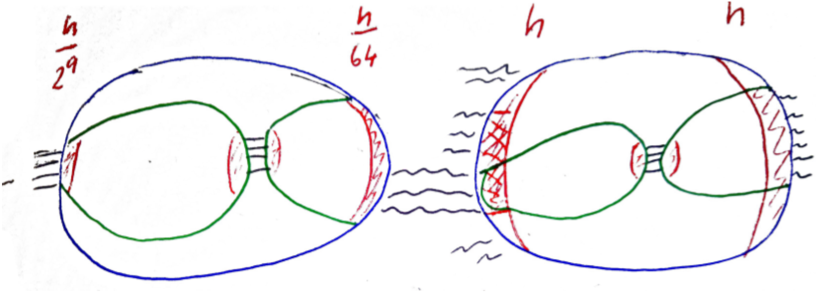
Implies Doubling theorem:



Theorem

Every good semi-brick has a splintering.

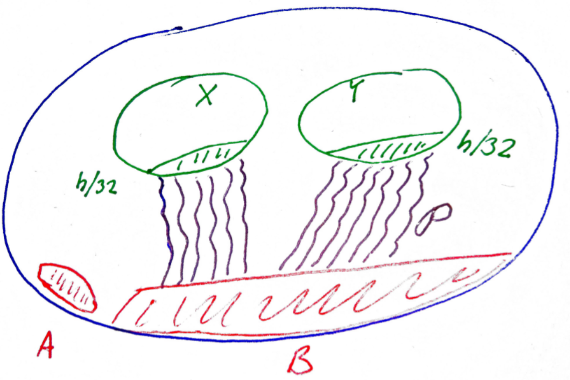
Implies Doubling theorem:



Definition

A weak splintering of a semi-brick (G, A, B) of height h:

- X and Y disjoint induced subgraphs of $G - (A \cup B)$.
- $\mathcal{P}$ a $(B - X \cup Y)$-linkage, $h/32$ paths to X and $h/32$ to Y.
- ends of $\mathcal{P}$ in X and Y are $(64, h/512)$-well-linked.



Lemma

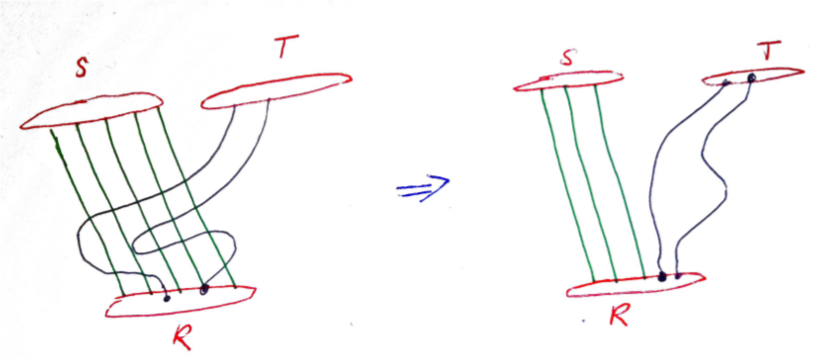
A weak splintering implies a splintering.

Cleaning lemma

Lemma

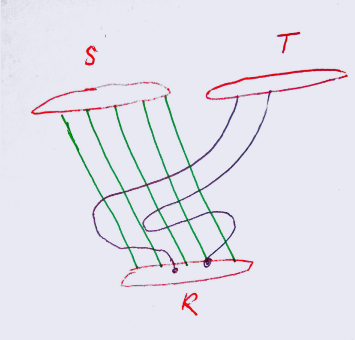
$\mathcal{P}_1$ an $(R - S)$-linkage of size a_1, an $(R - T)$ linkage of size $a_2 \le a_1 \Rightarrow$ an $(R - S \cup T)$-linkage $\mathcal{P}$ of size a_1 such that

- *$a_1 - a_2$ of the paths of $\mathcal{P}$ belong to $\mathcal{P}_1$,*
- *the remaining a_2 paths end in T.*



Proof.

- G minimal containing $\mathcal{P}_1$ and an $(R - T)$ linkage $\mathcal{P}_2$ of size a_2, ending in T_0
- augmenting path algorithm starting from $\mathcal{P}_2$ gives $\mathcal{P}$
- paths not to T_0 belong to $\mathcal{P}_1$



□

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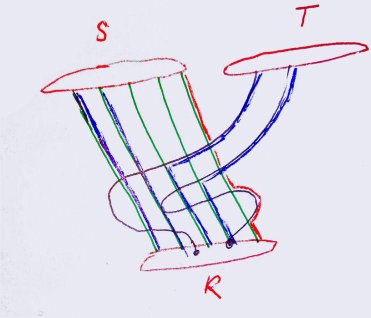
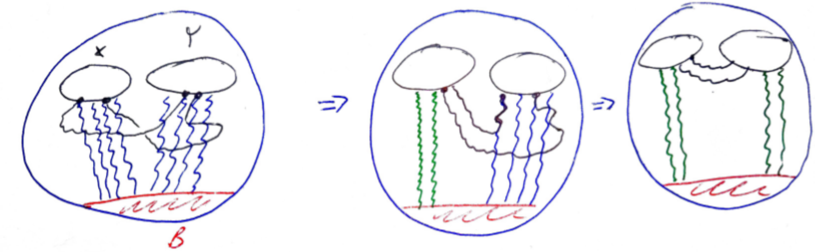
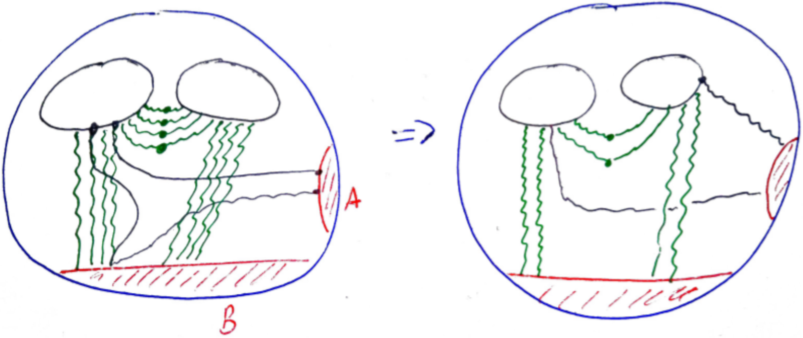


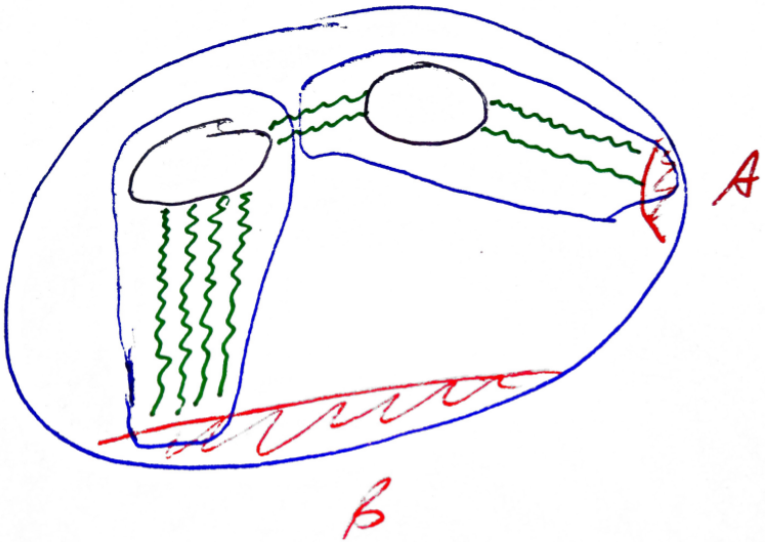
□

Lemma

A weak splintering implies a splintering.




A
B

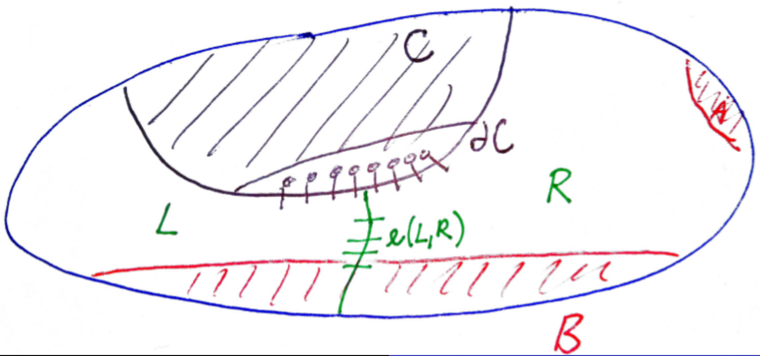


A
B

Definition

A cluster in a good semi-brick (G, A, B) is $C \subset G - (A \cup B)$ s.t. each vertex of C has at most one neighbor outside.
(a, k)-well-linked if ∂C is (a, k)-well-linked in C.

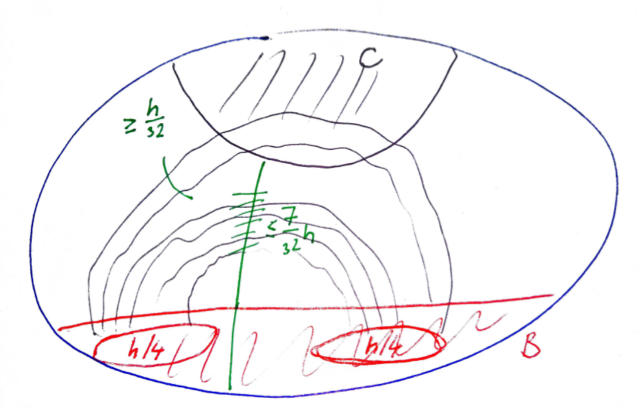
A balanced C-split: an ordered partition (L, R) of $V(G) \setminus V(C)$ such that $|R \cap B| \geq |L \cap B| \geq |B|/4$
$e(L, R)$ = number of edges from L to R.

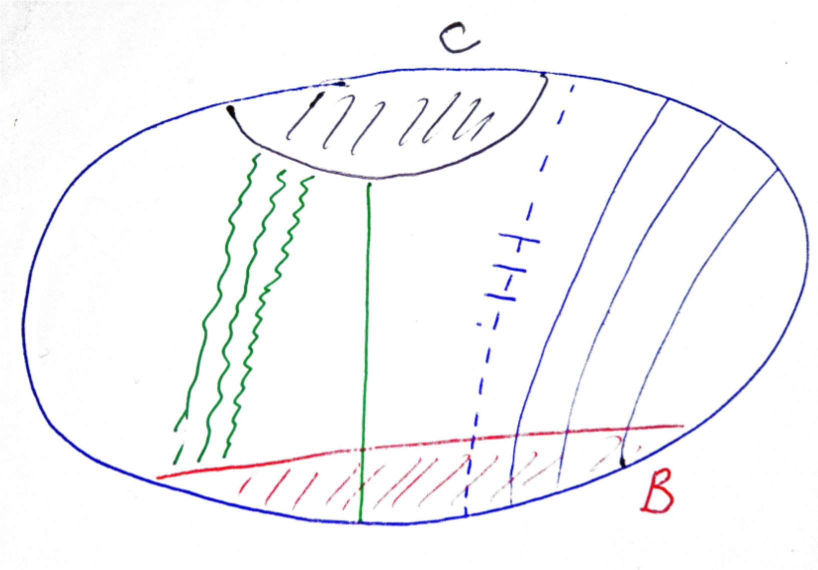


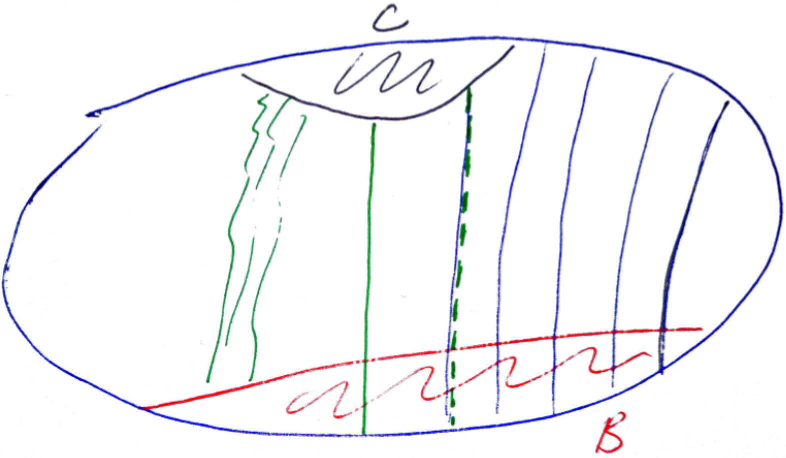
A balanced C-split (L, R) is good if $e(L, R) \leq \frac{7}{32}h$, perfect if additionally $\frac{1}{28}h \leq e(L, R)$.

Lemma

(G, A, B) a good semi-brick, C a perfect $(64, h/512)$-well-linked cluster, $|\partial C| \leq |A| + |B| \Rightarrow (G, A, B)$ contains a weak splintering.




C
B


c
m
B

Theorem

(G, A, B) a good semi-brick, C a good 23-well-linked cluster s.t. $|\partial C|$ is minimum and subject to that $|C|$ is minimum. Then either C is perfect or (G, A, B) contains a splintering.

Such C exists and $|\partial C| \leq |A| + |B|$: Consider $G - (A \cup B)$.

Important ideas:

- Looms (and especially planar looms) can be cleaned to grids.
- Path-of-sets systems and their doubling.
- Bounding the maximum degree, flows imply linkages.
- Cleaning lemma.