Recall the following definitions and results from the last lesson.

A set $W \subseteq V(G)$ is *a-well-linked* in a graph G if for all disjoint subsets A and B of W of the same size, G contains a flow from A to B of size $|A|$ and edge congestion at most a. We will also say W is (a,k)-well-linked if this holds for subsets of size at most k. We say W is *node-well-linked* if G contains a total $(A-B)$-linkage for any such subsets A and B. We say that two disjoint sets X and Y in G are *a-linked/node-linked* if for all $A \subseteq X$ and $B \subseteq Y$ of the same size, G contains a a flow from A to B of size $|A|$ and edge congestion at most a/a total $(A-B)$-linkage.

A *brick* of height h is a triple (G, A, B), where G is a graph and A and B are disjoint subsets of vertices of G of size h. The brick is *a-well-linked* if $A \cup B$ is a-well-linked in G, and *node-linked* if A and B are node-well-linked in G and A and B are node-linked in G. A *path-of-sets system* of width w and height h in a graph G is a sequence (H_1, A_1, B_1), ..., (H_w, A_w, B_w) of vertex-disjoint bricks of height h such that H_i is an induced subgraph of G for $i \in \{1, \ldots, w\}$, and G contains total $(B_i - A_{i+1})$-linkages $\mathcal{L}_i$ for $i \in \{1, \ldots, w-1\}$ such that the paths in $\bigcup_{i=1}^{w-1} \mathcal{L}_i$ are pairwise disjoint and disjoint from $H_1 \cup \ldots \cup H_w$ except for their endpoints; we say $\mathcal{L}_i$ is an *i-connector* of the system. The system is a-well-linked or node-linked if its bricks have these properties.

In a graph of bounded maximum degree, we can turn an a-well-linked path-of-sets system into a node-linked one.

Lemma 1. *Suppose (H_1, A_1, B_1), ..., (H_w, A_w, B_w) is an a-well-linked path-of-sets system of height at least $16(\Delta a+1)^2 h$ in a graph G of maximum degree at most Δ. Then there exist sets $A'_i \subseteq A_i$ and $B'_i \subseteq B_i$ of size h such that (H_1, A'_1, B'_1), ..., (H_w, A'_w, B'_w) is a node-linked path-of-sets system.*

We also argued that it suffices to find a sufficiently large and linked path-of-sets system.

Corollary 2. *If G has maximum degree Δ and contains an a-well-linked path-of-sets system of width $2n^2$ and height $32(\Delta a+1)^2 n(6n+9)$, then $W_n \preceq G$.*

It will be convenient to work with graphs of maximum degree three, as in these graphs edge-disjointness and vertex-disjointness of paths (almost) coincides. In the homework assignment, we have seen that it is possible to decrease the maximum degree to four while decreasing the treewidth only polynomially. The argument also gives a large node-well-linked sets of vertices. In fact, it is possible (by a significantly more difficult argument) to decrease the degree to three and lose only a polylogarithic fraction of the treewidth.

Theorem 3 (Chekuri and Chuzhoy [1])**.** *Every graph of treewidth t has a subgraph G of maximum degree at most three containing a node-well-linked set of $\Omega(t/\text{polylog } t)$ vertices of degree one.*

Let us remark that it would be possible to make the argument below work for graphs of maximum degree at most four, but at the cost of some complications. We say that a path-of-sets system has *maximum degree three* if the graph consisting of the union of the bricks and connectors of the system has maximum degree at most three, and the vertices of A_1 and B_w have degree one. Hence, Theorem 3 can be restated as saying that every graph of treewidth t contains a node-linked path-of-sets system of maximum degree three, width 1, and height $h = \Omega(t/\text{polylog } t)$.

1 Splintering bricks and doubling the system

The grid theorem follows from the fact that we can double the width of a path-of-sets system at the expense of decreasing its height by a constant factor.

Theorem 4. *If G contains a node-linked path-of-sets system of maximum degree three, width w and height h, then G also contains a 64-well-linked path-of-sets system of maximum degree three, width $2w$ and height $h/2^9$.*

Indeed, we can combine this theorem with Lemma 1 to turn the obtained path-of-sets system into node-linked one of height h/c for some fixed constant c. We then iterate the process $\log_2(2n^2)$-times, so that the resulting system has width $2n^2$ and height

$$\frac{h}{c^{\log_2(2n^2)}} = \frac{h}{(2n^2)^{\log_2 c}} = \frac{h}{\text{poly}(n)}.$$

For h sufficiently large (but just polynomial) in n, this enables us to apply Corollary 2 and obtain W_n as a minor.

To prove Theorem 4, we need to show how to locally split a brick into two. So that we can apply this splitting operation along the path-of-sets system, we need a variation on the definition of a brick. A *good semi-brick* of height h is a triple (G, A, B) with A and B disjoint sets of vertices of G, where

- G has maximum degree three and vertices of A and B have degree 1,
- $|A| = h/64$ and $|B| = h$,

- B is node-well-lined and A and B are node-linked.

A *splintering* of a semi-brick of height h is a pair of disjoint subgraphs X and Y of G and disjoint sets $A', C \subset V(X)$ and $D, B' \subset V(Y)$ such that

- $A' \subset A$ has size $h/512$ and $B' \subset B$ has size $h/64$,
- C and D have size $h/512$ and G contains a perfect matching between them, and
- $A' \cup C$ is 64-well-linked in X and $D \cup B'$ is $(64, h/512)$-well-linked in Y.

Lemma 5. *Every good semi-brick of height h has a splintering.*

It is easy to see Lemma 5 implies Theorem 4. Clearly, node-linkedness is stronger than the linkedness properties required for a semi-brick. We proceed along the path-of-sets system, applying Lemma 5. Suppose we already splintered the previous brick $(H_{i-1}, A_{i-1}, B_{i-1})$, and let $B'_{i-1} \subset B_{i-1}$ be the resulting subset of size $h/64$. We restrict A_i to the $h/4$ vertices connected to B'_{i-1} by the $(i-1)$-connector, turning the i-th brick into a good semi-brick, and apply Lemma 5 to split the semi-brick into X_i and Y_i, obtaining subsets $A'_i \subset A_i$ and $B'_i \subset B_i$. We restrict B'_{i-1} to the vertices connected to A'_i by the $(i-1)$-connector, turning Y_{i-1} into a 64-well-linked brick of height $h/512$. We repeat this procedure until the whole system consists of twice as many 64-well-linked bricks of height $h/512$.

Therefore, we only need to prove Lemma 5. Towards this goal, let us introduce a simpler object, a *weak splintering* $(X, Y, S_X, S_Y, \mathcal{P}_X, \mathcal{P}_Y)$, where X and Y are disjoint subgraphs of $G - (A \cup B)$, $\mathcal{P}_X$ and $\mathcal{P}_Y$ are disjoint $(B - S_X)$ and $(B - S_Y)$ linkages in G, each of size $h/32$ and only intersecting X and Y in their endpoints, S_X is $(64, h/512)$-well-linked in X and S_Y is $(64, h/512)$-well-linked in Y. We now aim to turn a weak splintering into a splintering. To this end, we repeatedly use the following Cleaning Lemma.

Lemma 6. *Let R, S, and T be sets of vertices in a graph G, and let $\mathcal{P}_1$ be an $(R - S)$-linkage of size a_1. Suppose also G contains an $(R - T)$ linkage of size $a_2 \leq a_1$. Then G contains an $(R - S \cup T)$-linkage $\mathcal{P}$ of size a_1 such that $a_1 - a_2$ of the paths of $\mathcal{P}$ belong to $\mathcal{P}_1$ and the remaining a_2 paths end in T.*

Proof. Let G' be a minimal subgraph of G that contains $\mathcal{P}_1$ as well as an $(R - T)$ linkage $\mathcal{P}_2$ of size a_2. Let T' be the set of ends of paths of $\mathcal{P}_2$ in T. Since G' contains an $(R - S \cup T)$-linkage of size a_1 (namely $\mathcal{P}_1$), the standard augmenting path flow algorithm started from $\mathcal{P}_2$ implies G' also contains such a linkage $\mathcal{P}$ where the endpoints include T'. We claim that all paths from $\mathcal{P}$ not ending in T' belong to $\mathcal{P}_1$; if not, one of these paths contains an edge e not belonging to $\mathcal{P}_1$. But then $G' - e$ contains $\mathcal{P}_1$ as well as an $(R - T')$ linkage of size a_2, contradicting the choice of G'. □

Lemma 7. *If a good semi-brick* (G, A, B) *contains a weak splintering* $(X, Y, S_X, S_Y, \mathcal{P}_X, \mathcal{P}_Y)$, *then it also contains a splintering.*

Proof. First, we construct a large (S_X, S_Y)-linkage while sacrificing a small part of $\mathcal{P}_X$ and $\mathcal{P}_Y$. Let B_X and B_Y be the ends of paths of $\mathcal{P}_X$ and $\mathcal{P}_Y$ in B. Let $k = h/64$, so that $|B_X| = |B_Y| = 2k$. Since B is node-well-linked, we can find a flow with vertex congestion 1 and size $2k$ from B_X to B_Y, and combining it with paths from $\mathcal{P}_X$ and $\mathcal{P}_Y$, we obtain a flow with vertex congestion 2 and size $2k$ from S_X to S_Y. This flow gives an $(S_X - S_Y)$-linkage of size k. From this linkage, we select a sublinkage $\mathcal{Q}_0$ of size $3k/4$.

We now apply the Cleaning Lemma to $\mathcal{P}_X$ and $\mathcal{Q}_0$ in the subgraph induced by these two linkages, obtaining $\mathcal{P}'_X \subset \mathcal{P}_X$ of size $5k/4$ and a disjoint (S_X, S_Y)-linkage $\mathcal{Q}'_0$ of size $3k/4$. Then, we apply the Cleaning Lemma to $\mathcal{P}_Y$ and $\mathcal{Q}'_0$, obtaining $\mathcal{P}'_Y \subset \mathcal{P}_Y$ of size $5k/4$ and a disjoint (S_X, S_Y)-linkage $\mathcal{Q}$ of size $3k/4$.

Next, we find a large $(A, S_X \cup S_Y)$-linkage while sacrificing small parts of $\mathcal{P}'_X$, $\mathcal{P}'_Y$, and $\mathcal{Q}$. Since A and B are node-linked, there exists a flow of size $2k$ from a subset of A of size $2k$ to B_X and node congestion 1, and combining it with $\mathcal{P}_X$, we obtain a flow with node congestion 2 to $S_X \cup S_Y$. As before, this gives an $(A, S_X \cup X_Y)$-linkage $\mathcal{R}_0$ of size $k/4$.

We select one vertex on each path of $\mathcal{Q}$, forming a set Z (we insert auxiliary fake vertices on those paths that are just single edges), so we can view $\mathcal{Q}$ as a $(Z - S_X \cup S_Y)$-linkage. We again apply the Cleaning Lemma to $\mathcal{P}'_X \cup \mathcal{P}'_Y \cup \mathcal{Q}$ and $\mathcal{R}_0$, obtaining $\mathcal{L} \subset \mathcal{P}'_X \cup \mathcal{P}'_Y \cup \mathcal{Q}$ of size $|\mathcal{P}'_X \cup \mathcal{P}'_Y \cup \mathcal{Q}| - k/4$ and a disjoint $(A, S_X \cup X_Y)$-linkage $\mathcal{R}$ of size $k/4$. From $\mathcal{L}$, we can select $\mathcal{P}''_X \subset \mathcal{P}'_X$ of size $|\mathcal{P}'_X| - k/4 = k$, $\mathcal{P}''_Y \subset \mathcal{P}'_Y$ of size k, and $\mathcal{Q}'$ of size $k/8 \leq |\mathcal{Q}| - k/4$.

By symmetry, we can assume that $k/8 = h/512$ paths $\mathcal{R}_X$ from $\mathcal{R}$ end in S_X. We can combine $\mathcal{R}_X$, X, and all but one edge of every path from $\mathcal{Q}'$ to form the left part of the splintering, and Y together with $\mathcal{P}''_Y$ to form the right part of the splintering. □

2 Weak splinterings from perfect clusters

Consider a good semi-brick (G, A, B). A *cluster* is an induced subgraph C of $G - (A \cup B)$. For a cluster C, we let ∂C denote the set of vertices of C

incident with edges outside of C. We say that C is (a,k)-*well-linked* if ∂C is (a,k)-well-linked in C. A *balanced* C-*split* is an ordered partition (L,R) of $V(G) \setminus V(C)$ such that $|R \cap B| \geq |L \cap B| \geq |B|/4$; it is *minimum* if the number $e(L,R)$ of edges of G between L and R is minimum among all balanced C-splits. We say that C is *good* if $e(L,R) \leq \frac{7}{32}h$, and perfect if additionally $\frac{1}{28}h \leq e(L,R)$. A perfect, sufficiently linked cluster gives a weak splintering.

Lemma 8. *Let* (G,A,B) *be a good semi-brick and let* C *be a perfect* $(64,h/512)$*-well-linked cluster. If* $|\partial C| \leq |A|+|B|+1$ *and each vertex of* C *has at most one neighbor outside of* C*, then* (G,A,B) *contains a weak splintering.*

Proof. Let (L,R) be a minimum balanced C-split. Since B is $(1,h/4)$-well-linked, G contains $h/4$ vertex-disjoint paths from $L \cap B$ to $R\cap$, and at least $h/4 - e(L,R) \geq h/32$ of them hits C; let $\mathcal{P}_C$ consist of the initial segments of these paths until they hit C, and let S_C be the set of their ends in C.

Note also this implies at least $h/4 - e(L,R)$ edges from C are incident with L, and thus the number of edges leaving R is at most $e(L,R)+e(C,R) = e(L,R)+|\partial C|-e(C,L) \leq |A|+|B|+1+2e(L,R)-h/4 = \frac{49}{64}h+2e(L,R)+1$. We now perform the following algorithm, refining a partition $\mathcal{P}$ of R; initially, we set $\mathcal{P} = \{R\}$. As long as there exists a (necessarily unique) part $Y' \in \mathcal{P}$ such that $|Y' \cap B| > |R \cap B|/2$, we check whether $\partial Y' \cup (Y' \cap (A \cup B))$ is $(64,h/512)$-well-linked in $G[Y']$. If so, we let $Y = G[Y' \setminus (A \cup B)]$; the cluster Y is also $(64,h/512)$-well-linked, since all vertices of $A \cup B$ have degree one. And, we let $\mathcal{P}_Y$ consist of $h/32$ edges between B and Y, and let S_Y denote their ends in Y. Then $(C,Y,S_C,S_Y,\mathcal{P}_C,\mathcal{P}_Y)$ is a weak splintering. Otherwise, $Y' = Y_1 \cup Y_2$ for disjoint Y_1 and Y_2 such that $e(Y_1,Y_2) < \frac{1}{64}\min(|Y_1|,|Y_2|)$; we replace Y' by Y_1 and Y_2 in $\mathcal{P}$.

Suppose this continued till every part would contain at most half of the vertices of $|R \cap B|$. A technical calculation (which we skip) shows that we did not create too many edges between different parts of $\mathcal{P}$—less than $e(L,R)/2$ in total (this calculation uses the fact that $e(L,R) \geq h/28$). For each part $P \in \mathcal{P}$, let $i(P)$ denote the number of edges from P to the other parts of $\mathcal{P}$, and $o(P)$ the number of edges from P to L. Then

$$\sum_{P \in \mathcal{P}} (i(P) - o(P)) = \sum_{P \in \mathcal{P}} i(P) - \sum_{P \in \mathcal{P}} o(P) < 2 \cdot \frac{e(L,R)}{2} - e(L,R) = 0,$$

and thus there exists $P \in \mathcal{P}$ such that $i(P) < o(P)$. However, $|R \cap B \setminus P| \geq |R \cap B|/2 \geq |B|/4$, and thus $(L \cup P, R \setminus P)$ is a balanced C-split with $e(L,R) + i(P) - o(P) < o(L,R)$ edges between the parts, contradicting the minimality of (L,R). $\square$

The proof of the grid theorem thus would be finished by proving the following claim.

Theorem 9. *Let (G, A, B) be a good semi-brick and let C be a good 23-well-linked cluster such that $|\partial C|$ is minimum and subject to that $|C|$ is minimum. Then either C is perfect or (G, A, B) contains a splintering.*

Indeed, note that $G - (A \cup B)$ is a good 23-well-linked cluster, and thus $|\partial C| \leq |A| + |B|$. Furthermore, it is easy to see that the minimality of $|C|$ and the fact that G has maximum degree at most three implies that each vertex of C has at most one neighbor outside. Therefore, if C turns out to be perfect, we obtain a splintering by Lemma 8.

The proof of Theorem 9 is still quite technical, and we opt to stop our exposition here.

References

[1] C. Chekuri and J. Chuzhoy, *Degree-3 treewidth sparsifiers*, in Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms, SIAM, 2015, pp. 242–255.