The grid theorem

Theorem

If $tw(G) \geq f(n)$, then $W_n \preceq G$.

Upper bounds:

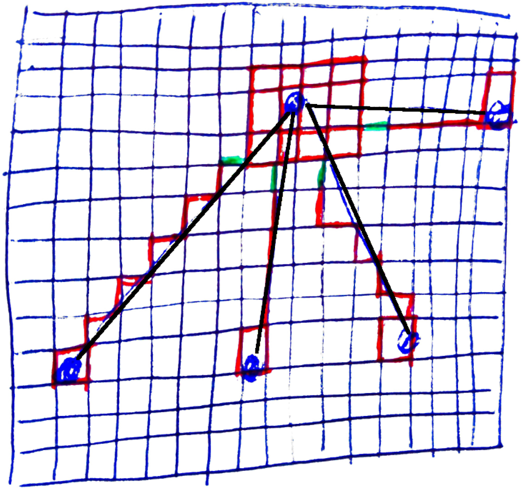
- f exists: Robertson and Seymour'84
- $f(n) \leq 20^{2n^5}$: Robertson, Seymour, and Thomas'94
- $f(n) = O(n^{100})$: Chekuri, Chuzhoy'16
- $f(n) = O(n^9 \text{polylog } n)$: Chekuri, Tan'19

Lower bounds:

- $f(n) = \Omega(n^2)$ because of K_n
- $f(n) = \Omega(n^2 \log n)$ because of random graphs

Lemma

For every planar graph H, there exists n_H such that $H \preceq W_{n_H}$.



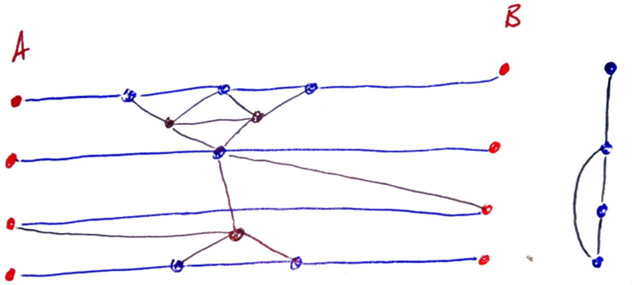
Forbidding a planar graph

Corollary

For H planar, if G does not contain H as a minor, then $tw(G) < f(n_H)$.

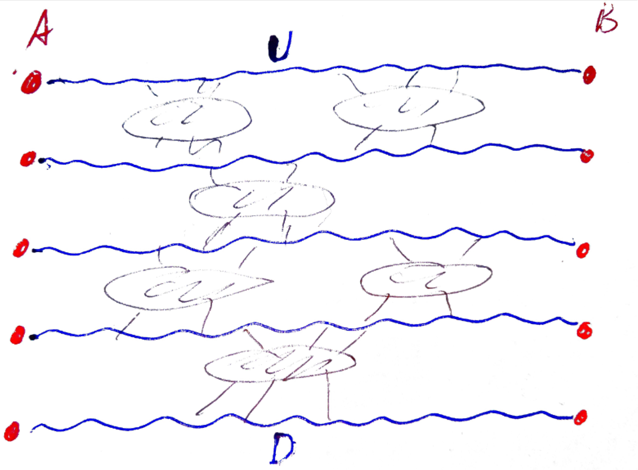
Definition

$(A - B)$-linkage: Set $\mathcal{L}$ of disjoint $A - B$ paths.
Total if $|A| = |B| = |\mathcal{L}|$.
$G_{\mathcal{L}}$: $L_1, L_2 \in \mathcal{L}$ adjancent if G contains a path from L_1 to L_2 disjoint from rest of $\mathcal{L}$.



Definition

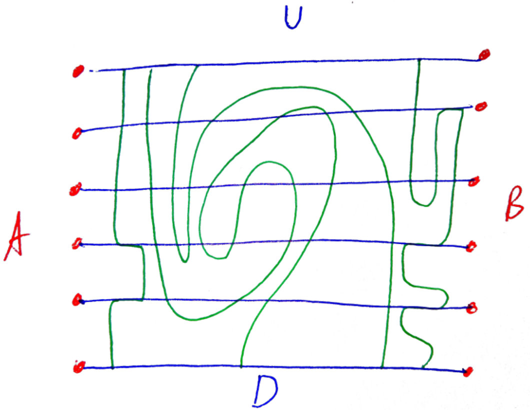
Loom (G, A, B, U, D) of order $|A| = |B|$: For every total $(A - B)$-linkage containing U and D, $G_{\mathcal{L}}$ is a path from U to D.



From looms to grids

Theorem

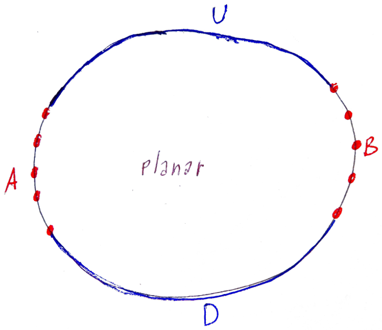
Loom (G, A, B, U, D) *of order* $n+2$, $\exists$ *a total* $(A-B)$*-linkage containing* U *and* D, *a* $(V(U)-V(D))$*-linkage of size* $n \Rightarrow W_n \preceq G$.



Planar looms

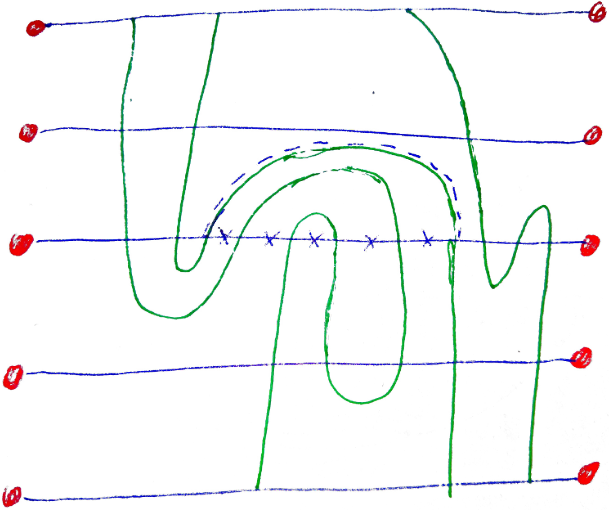
Definition

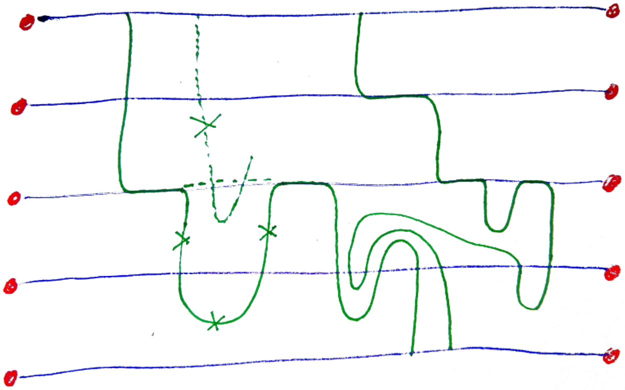
A loom (G, A, B, U, D) is <u>planar</u> if G is a plane graph and A, U, B, D appear in the boundary of the outer face in order.

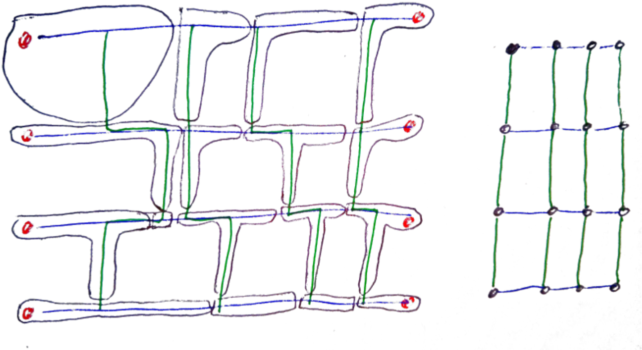


Lemma

The theorem holds for a planar loom of order n.



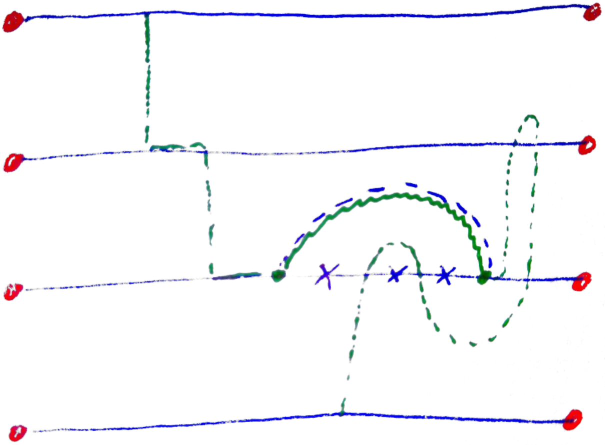


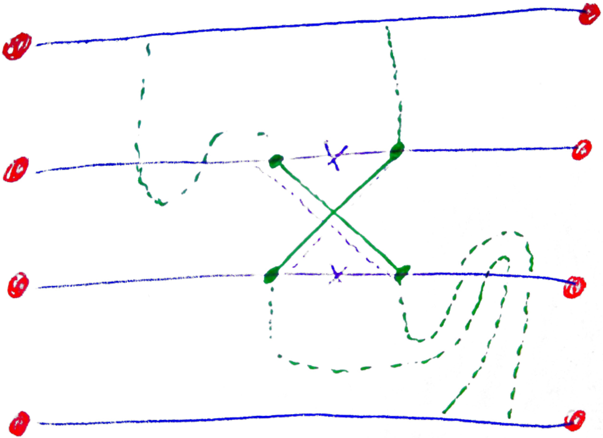


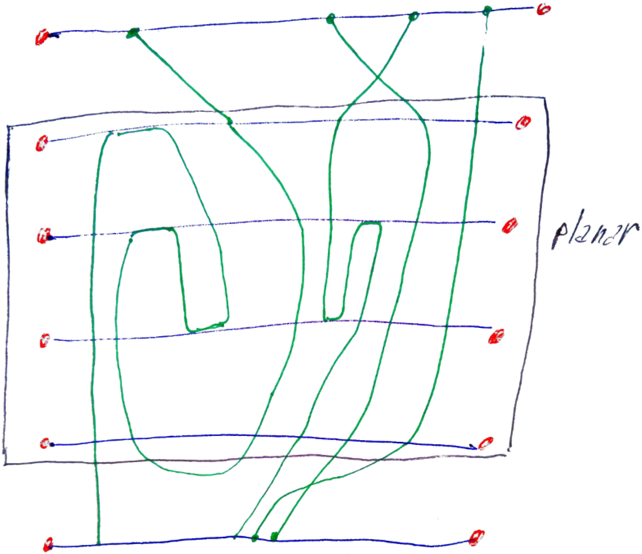
Planarizing a loom

Lemma

Loom of order $n+2$ + linkages $\Rightarrow$ planar loom of order n + linkages.



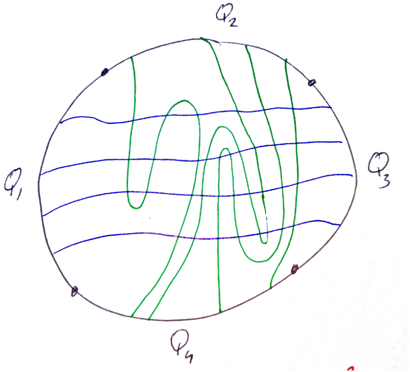



planar

Remark on planar graphs

Corollary

G plane, outer face bounded by cycle $C = Q_1 \cup \ldots \cup Q_4$*, exists a* $(V(Q_1) - V(Q_3))$*-linkage and a* $(V(Q_2) - V(Q_4))$*-linkage of order* $n \Rightarrow W_n \preceq G$.



Theorem

There exists $g(n) = O(n)$ s.t. G planar, $tw(G) \geq g(n) \Rightarrow W_n \preceq G$.

Proof.

Lecture notes, Theorem 6. □

Corollary

G planar $\Rightarrow tw(G) = O(\sqrt{|V(G)|}) \Rightarrow G$ contains a balanced separator of order $O(\sqrt{|V(G)|})$.

Definition

Disjoint sets A and B are node-linked if for all $W \subseteq A$ and $Z \subset B$ of the same size, there exists a total $(W - Z)$-linkage.

Definition

(G, A, B) a brick of height h if A, B disjoint and $|A| = |B| = h$.
Node-linked if A and B are node-linked.

Lemma

Connected graph with $\geq 2a(b+5)$ vertices contains either a spanning tree with $\geq a$ leaves, or a path of b vertices of degree two.

Proof.

Lecture notes, Lemma 11. □

Lemma

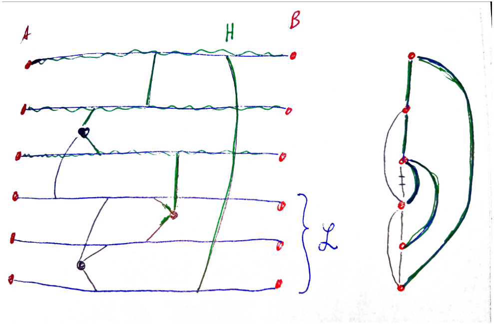
(G, A, B) a node-linked brick of height $2n(6n+9)$, $W_n \not\preceq G \Rightarrow$ an $(A-B)$-linkage $\mathcal{L}$ of size n, a connected subgraph H disjoint from and with a neighbor in each path of $\mathcal{L}$.

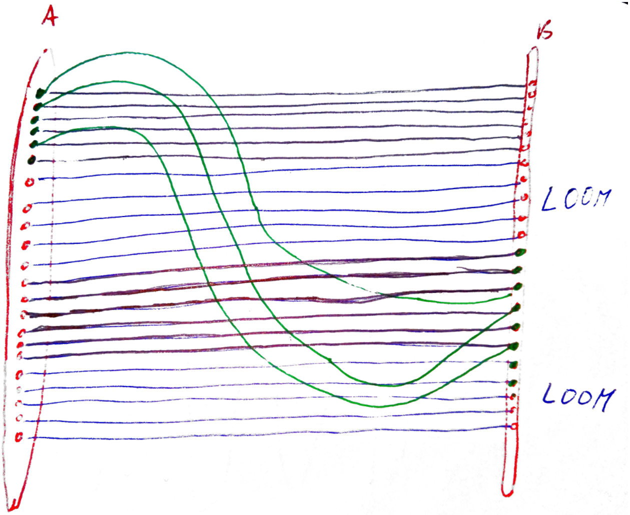
Proof.

$\mathcal{L}_0$: a total $(A-B)$-linkage s.t. $G_{\mathcal{L}_0}$ has smallest number of vertices of degree two.

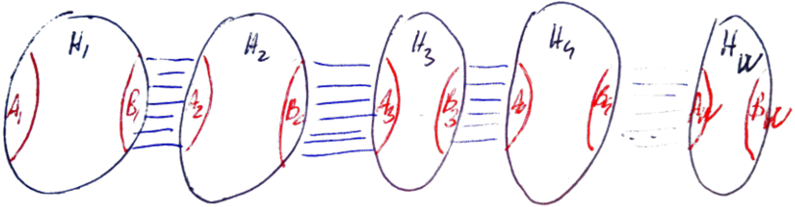
- Spanning tree with n leaves: gives H.
- Path of $6n+4$ vertices of degree two: next slide.

□



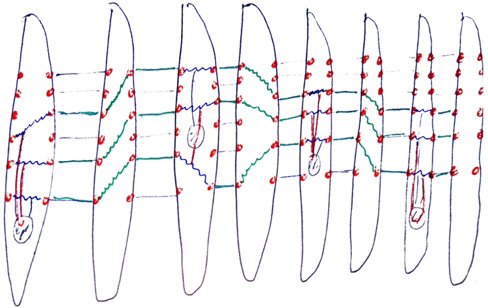

A
B
LOOM
LOOM

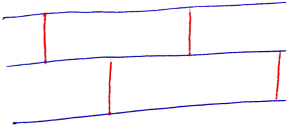
Path-of-sets system



Lemma

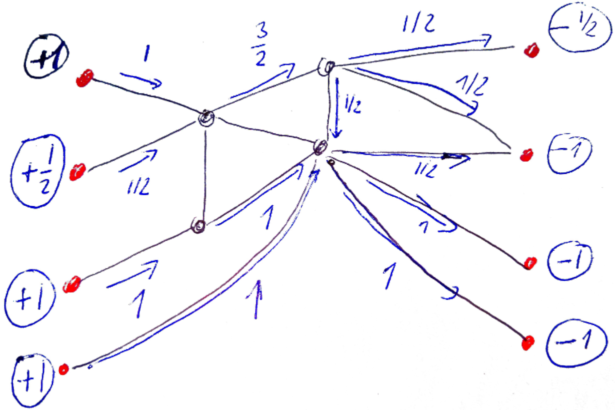
Node-linked path-of-sets system of width $2n^2$ and height $2n(6n+9)$, then $W_n \preceq G$.





Definition

Flow from A to B: Flow at most 1 starts in each vertex of A and ends in each vertex of B, no flow is created or lost elsewhere.
Edge/vertex congestion: maximum amount of flow over an edge/through a vertex.



Observation

Edge congestion a, maximum degree $\Delta \Rightarrow$ vertex congestion $\leq \Delta a + 1$.

Observation

Flow of size s and vertex congestion $c \Rightarrow$ flow of size s/c and vertex congestion $1 \Rightarrow (A - B)$-linkage of size $\geq s/c$.

Definition

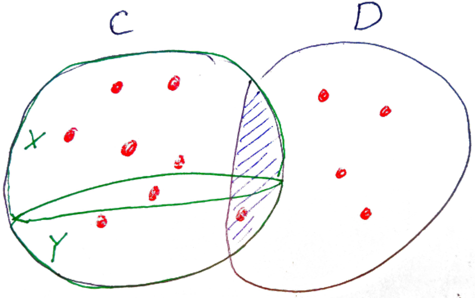
Set W is a-well-linked/node-well-linked if for all $A, B \subset W$ disjoint, of the same size, there exists a flow from A to B of size $|A|$ and edge congestion $\leq a$ / a total $(A - B)$-linkage.

Observation

- *Either W is a-well-linked, or there exists $X \subseteq V(G)$ such that $|\partial X| < a \min(|W \cap X|, |W \setminus X|)$.*
- *Either W is node-well-linked, or there exists a separation (X, Y) of G of order less than $\min(|W \cap V(X)|, |W \cap V(Y)|)$.*

Lemma

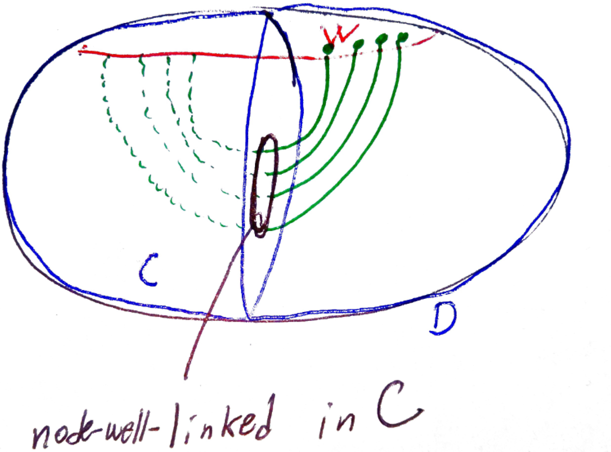
(C, D) a separation of minimum order such that $|V(C) \cap W|, |V(D) \cap W| \geq |W|/4$, $|V(C) \cap W| \geq |W|/2 \Rightarrow V(C \cap D)$ is node-well-linked in C.



$$o(C,D) \leq o(X, Y \cup D) = o(C,D) + o(X,Y) - |V(C \cap D) \cap Y|$$

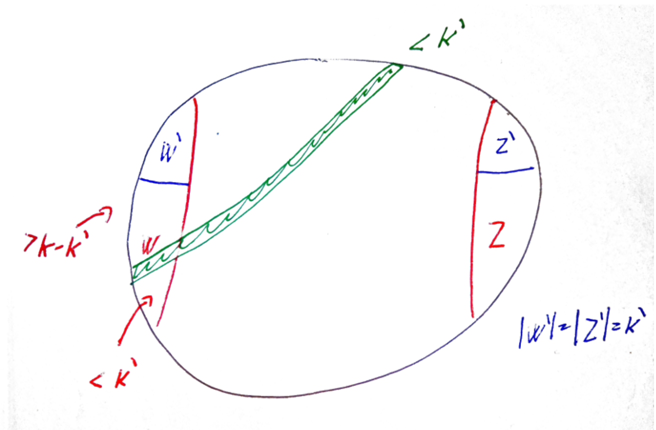
Lemma

W a-well-linked $\Rightarrow \exists\, W' \subseteq W$, $|W'| \geq \frac{|W|}{4(\Delta a+1)}$, W' *node-well-linked.*



Lemma

W and Z node-well-linked of size at least k, $W \cup Z$ is a-well-linked $\Rightarrow \forall\ W' \subset W$, $Z' \subset Z$, $|W'|, |Z'|, |W'| \leq \frac{k}{\Delta a+2}$, the sets W' and Z' are node-linked.

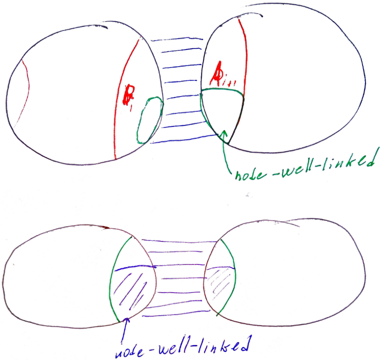


Definition

A path-of-sets system is *a*-well-linked if in each brick (H, A, B), the set $A \cup B$ is *a*-well-linked.

Lemma

a-well-linked path-of-sets system of height at least $16(\Delta a + 1)^2 h \Rightarrow$ *node-linked one of height h.*



Corollary

Maximum degree Δ, an a-well-linked path-of-sets system of width $2n^2$ and height $32(\Delta a+1)^2 n(6n+9) \Rightarrow$ a minor of W_n.

TODO:

- Graph of large treewidth has a subgraph of large treewidth and bounded maximum degree (homework assignment).
- Large treewidth $\Rightarrow$ large a-well-linked path-of-sets system (next lecture).