Let $f(a)$ be the minimum integer such that every graph of average degree at least $f(a)$ contains K_a as a minor.

Theorem (Thomasson)

$$f(a) = (0.638\ldots + o(1))a\sqrt{\log a}$$

Theorem (Norin and Thomas)

For every a there exists N such that every a-connected graph G with at least N vertices either

- *contains K_a as a minor, or*
- *is obtained from a planar graph by adding at most $a - 5$ apex vertices.*

We will prove a simpler result:

Theorem (Böhme, Kawarabayashi, Maharry and Mohar)

Every $(3a+2)$-connected graph of minimum degree at least $20a$ and with $\gg a, k, s, t$ vertices either

- *contains $sK_{a,k}$ as a minor, or*
- *contains a subdivision of $K_{a,t}$.*

Observation

$$K_a \preceq K_{a-1,a}$$

Corollary

Every $(3a-1)$-connected graph of minimum degree at least $20a$ and with $\gg a$ vertices contains K_a as a minor.

Corollary

Every $(3a+2)$-connected graph of minimum degree at least $20a$, maximum degree less than t, and with $\gg a, k, s, t$ vertices contains $sK_{a,k}$ as a minor.

Definition

A graph M is k-linked if for all distinct $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(G)$, M contains disjoint paths from s_1 to t_1, ..., s_k to t_k.

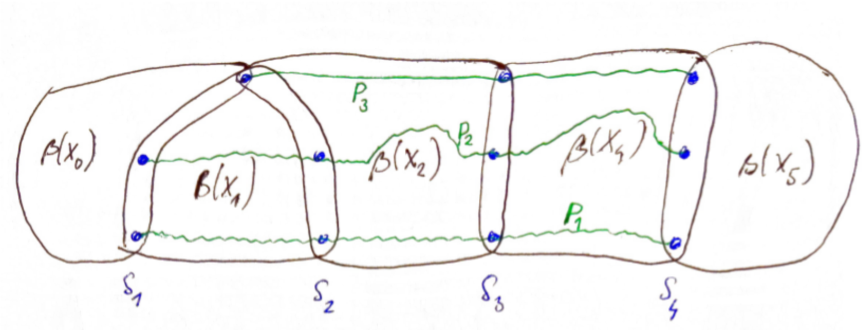
Theorem

A graph of average degree at least $13k$ contains a k-linked subgraph.

- A path decomposition $(x_0 x_1 \ldots x_m, \beta)$ of H.
- $S_i = \beta(x_{i-1}) \cap \beta(x_i)$.

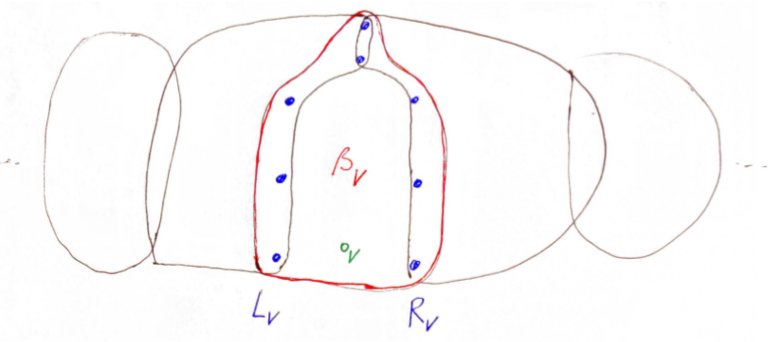
Definition

q-linked if $|S_1| = |S_2| = \ldots = |S_m| = q$ and H contains q vertex-disjoint linking paths from S_1 to S_m.



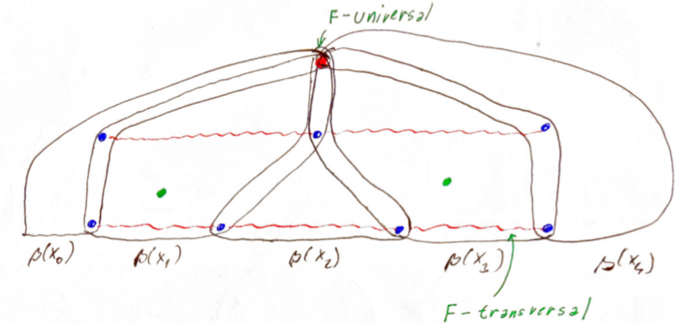
A vertex v is internal if it does not belong to $\beta(x_0) \cup \beta(x_m) \cup \bigcup_{i=1}^{m} S_i$.

- i_v: $v \in \beta(x_{i_v})$
- $\beta_v = \beta(x_{i_v})$, $L(v) = S_{i_v-1}$, $R_v = S_{i_v}$.
- Focus F: Set of internal vertices belonging to distinct bags.



A linking path P is

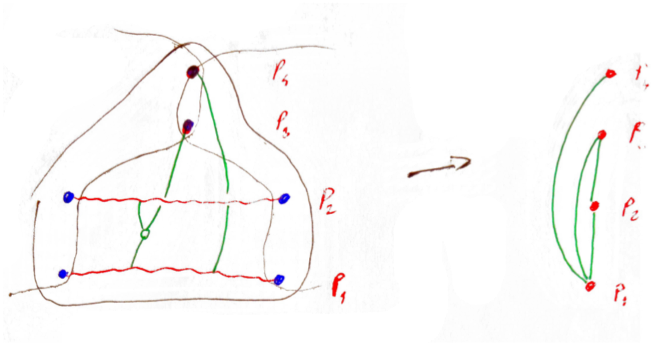
- <u>F-universal</u> if there exists $u_P \in V(P)$ such that $V(P) \cap \beta_v = \{u_P\}$ for all $v \in F$.
- <u>F-transversal</u> if $V(P) \cap \beta_v$ and $V(P) \cap \beta_{v'}$ are disjoint for all distinct $v, v' \in F$.



Observation

If $|F| \geq (\ell + 3)\ell$, then there exists $F' \subseteq F$ of size at least ℓ such that P is either F'-universal or F'-transversal.

For $v \in F$, let Γ_v be the graph with $V(\Gamma_v) = \{P_1, \ldots, P_q\}$, $P_iP_j \in E(\Gamma_v)$ iff $H[\beta_v]$ contains a path from P_i to P_j disjoint from all other linking paths.



Observation

If $|F| \gg \ell$, then there exists $F' \subseteq F$ of size at least ℓ such that $\Gamma_v = \Gamma_{v'}$ for all $v, v' \in F'$.

Lemma

Assume for every $v \in F$:

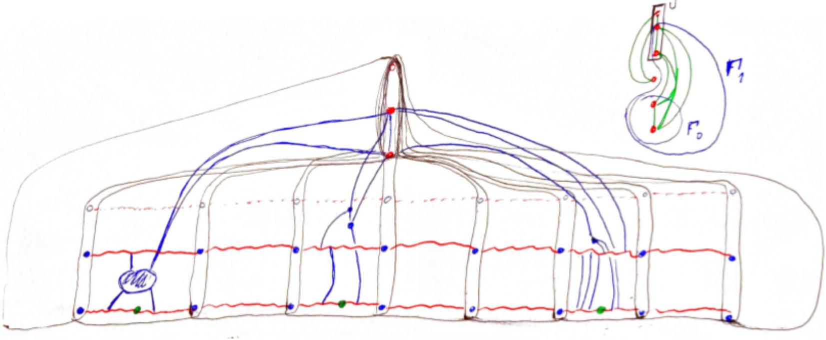
- *v lies on P_1.*
- *No separation (A, B) of $H[\beta_v]$ of order less than $3a + 2$ with $L_v \cup R_v \cup \{v\} \subseteq A$ and $B \not\subseteq A$.*
- *Vertices of $\beta_v \setminus (L_v \cup R_v)$ have degree at least $20a - 4$ in $H[\beta_v]$.*

If $|F| \gg a, k, s, t, q$, then H contains $sK_{a,k}$ as a minor or $K_{a,t}$ as a topological minor.

Assume uniformity;

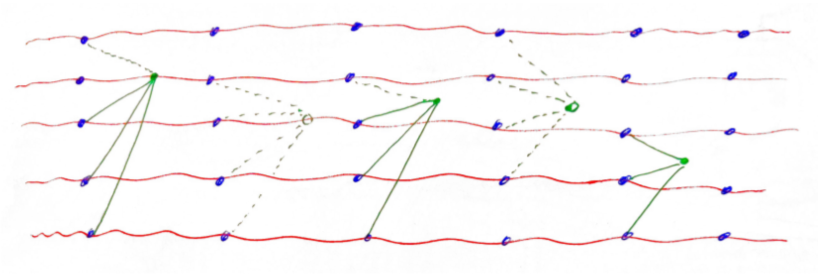
- U: F-universal paths (vertices).
- $\Gamma = \Gamma_v$ for $v \in F$.
- Γ_1: the U-bridge containing P_1, $\Gamma_0 = \Gamma_1 - U$.

- For each $v \in F$, let $H_1(v)$ be the U-bridge of $H[\beta_v]$ containing v.
    - $H_1(v)$ is intersected exactly by linking paths in Γ_1.
- Let H_1 consist of
    - linking paths in Γ_1 and
    - $H_1(v)$ for $v \in F$.
- Let $H_0(v) = H_1(v) - U$, $H_0 = H_1 - U$.



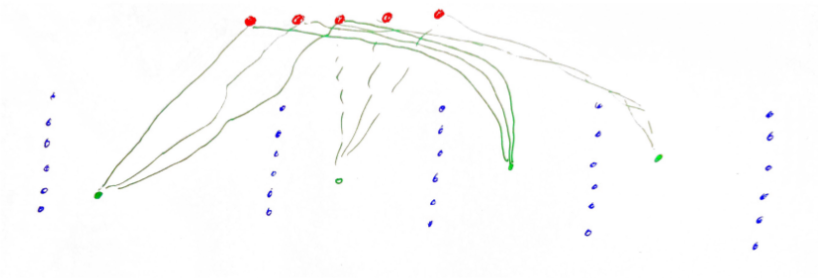
Observation

If for $sk\binom{q}{a}$ vertices $v \in F$, there exists $x \in V(H_0(v))$ with $\geq 2a+1$ neighbors in $(L_v \cup R_v) \setminus U$, then $sK_{a,k} \preceq H$.



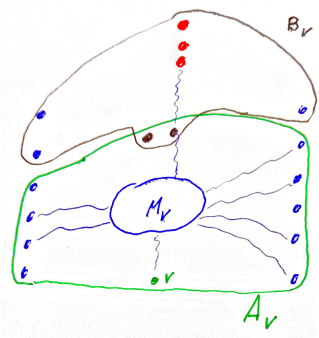
Observation

If for $t\binom{q}{a}$ vertices $v \in F$, there exist a disjoint paths from v to U in $H_1(v)$, then H contains a subdivision of $K_{a,t}$.



For all other $v \in F$:

- Separation (A_v, B_v) of $H_1(v)$ of order less than a with $v \in V(A_v) \setminus V(B_v)$ and $U \cap V(H_1) \subseteq V(B_v)$.
- $H[A_v - (\{v\} \cup L_v \cup R_v \cup V(B_v))]$ has minimum degree at least $17a - 5 \Rightarrow (a+1)$-linked subgraph M_v.
- By assumptions: $3a + 2$ disjoint paths from M_v to $L_v \cup R_v \cup \{v\}$.
    - $2a + 2$ end in $(L_v \cup R_v) \setminus U$.
    - Can be redirected so that $a + 1$ end in $X_v \subseteq L_v \setminus U$ and $a + 1$ in $Y_v \subseteq R_v \setminus U$.



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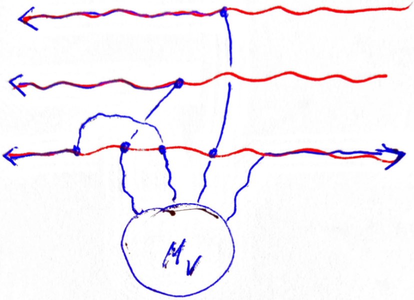

For all other $v \in F$:

- Separation (A_v, B_v) of $H_1(v)$ of order less than a with $v \in V(A_v) \setminus V(B_v)$ and $U \cap V(H_1) \subseteq V(B_v)$.
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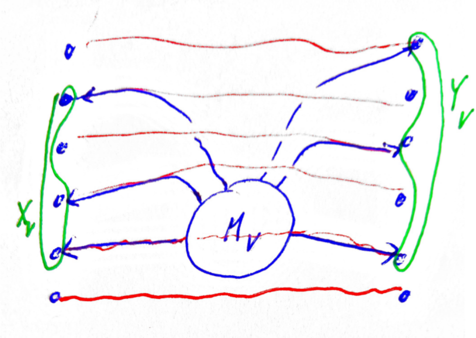
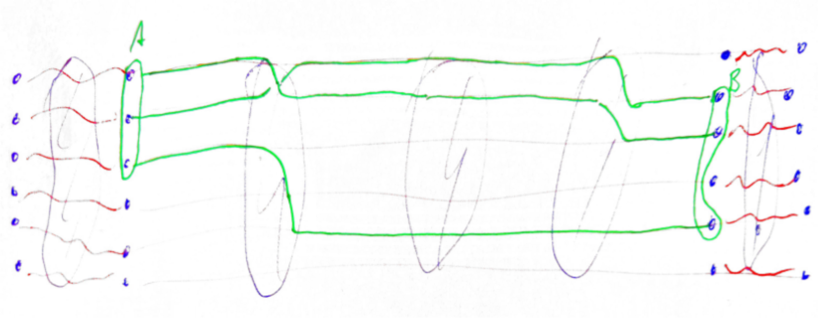
- By assumptions: $3a + 2$ disjoint paths from M_v to $L_v \cup R_v \cup \{v\}$.
  - $2a + 2$ end in $(L_v \cup R_v) \setminus U$.
  - Can be redirected so that $a + 1$ end in $X_v \subseteq L_v \setminus U$ and $a + 1$ in $Y_v \subseteq R_v \setminus U$.



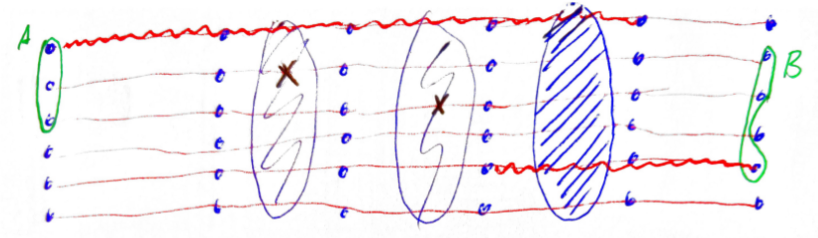
Observation

If there are at least $a+1$ vertices of F between v and v' and $A \subseteq R_v$ and $B \subseteq L_{v'}$ have size $a+1$, then H_0 contains $a+1$ disjoint paths from A to B.

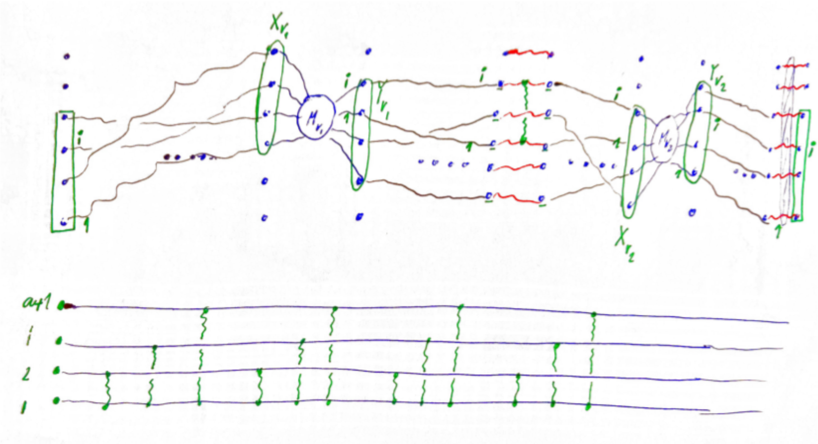


Observation

If there are at least $a + 1$ vertices of F between v and v' and $A \subseteq R_v$ and $B \subseteq L_{v'}$ have size $a + 1$, then H_0 contains $a + 1$ disjoint paths from A to B.



$sK_{a,k} \preceq H.$



A tree decomposition (T, β) of a graph G is

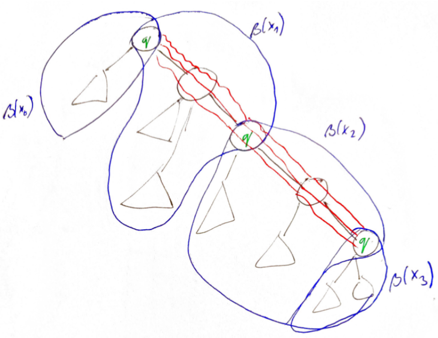
- linked if for any $x, y \in V(T)$ and an integer k, either
  - G contains k vertex-disjoint paths from $\beta(x)$ to $\beta(y)$, or
  - there exists $z \in V(T)$ separating x from y in T such that $|\beta(z)| < k$.
- nondegenerate if no two bags are the same.

Theorem (Thomas)

Every graph G has a nondegenerate linked tree decomposition of width $tw(G)$.

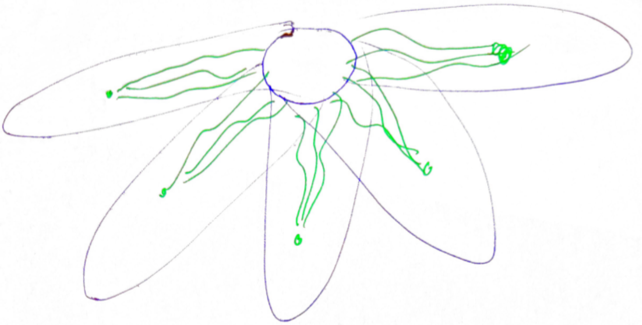
Lemma

Every $(3a+2)$*-connected graph of minimum degree at least* $20a$*, treewidth at most* q *and with* $\gg a, k, s, t, q$ *vertices either contains* $sK_{a,k}$ *as a minor or* $K_{a,t}$ *as a topological minor.*

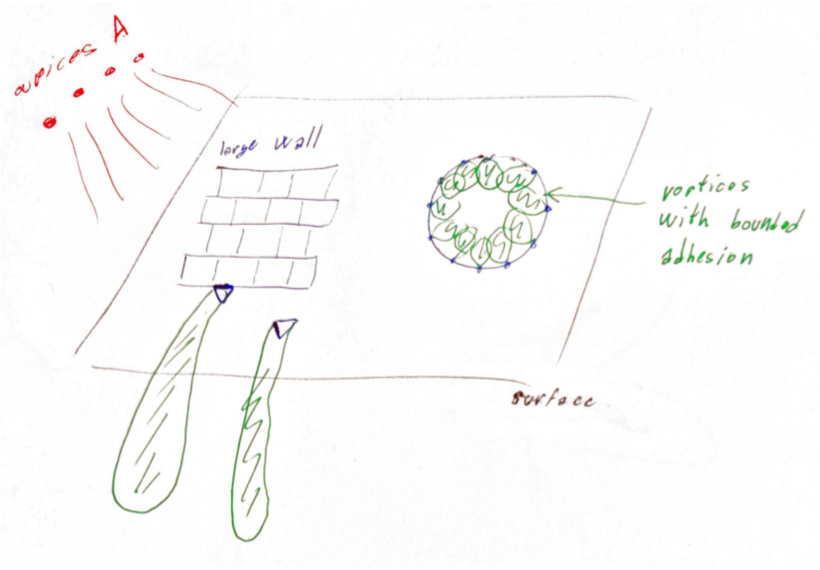


Lemma

Every $(3a+2)$-connected graph of minimum degree at least $20a$, treewidth at most q and with $\gg a, k, s, t, q$ vertices either contains $sK_{a,k}$ as a minor or $K_{a,t}$ as a topological minor.

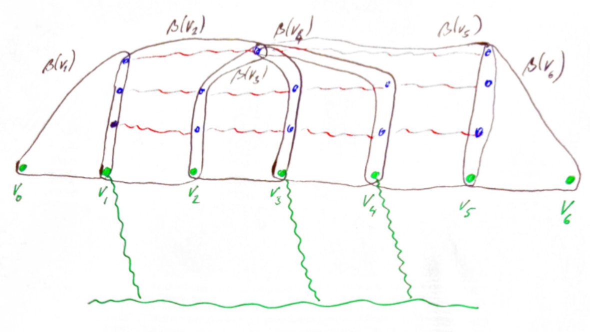


Local structure decomposition with a wall:



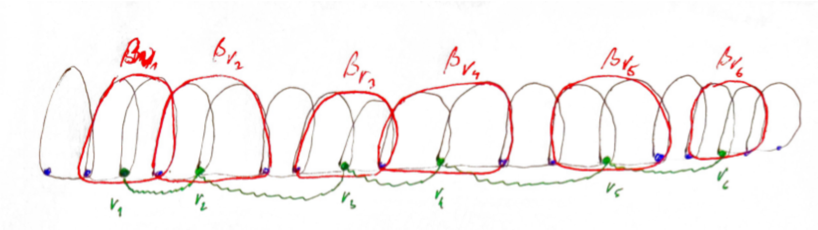
Apex vertices A, a vortex R with boundary $\partial R = v_0 v_1 \dots v_m$.

- R is q-linked: its path decomposition $(v_1 \dots v_m, \beta)$ satisfies
  - $\beta(v_i) \cap \{v_0, \dots, v_m\} = \{v_{i-1}, v_i\}$ for $i = 1, \dots, m$,
  - $|\beta(v_i) \cap \beta(v_{i+1})| = q + 1$ for $i = 1, \dots, m-1$, and
  - $R - \partial R$ contains q paths from $\beta(v_1)$ to $\beta(v_m)$.
- $v \in \partial R$ is local if all but at most four neighbors of v belong to $V(A \cup R)$.
- $F \subseteq \partial R$ is attached to a comb if there exist paths outside of $R \cup A$ from F to a path, ending in order.



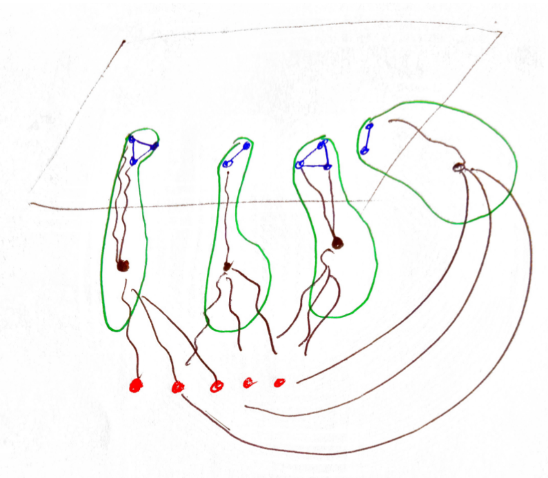
Lemma

$(3a+2)$-connected graph G of minimum degree at least $20a$, apices A, q-linked vortex R, $F \subseteq \partial R$ local vertices attached to a comb, $|F| \gg a, k, s, t, q, |A| \Rightarrow G$ contains $sK_{a,k}$ as a minor or $K_{a,t}$ as a topological minor.



- Many vertices with $\geq a$ neighbors in A, or
- many pieces attaching to the surface part

imply subdivision of $K_{a,t}$.



- Large treewidth $\Rightarrow$ large wall W.
- No $sK_{a,k}$-minor $\Rightarrow$ decomposition with respect to W;
  - we can assume the vortices are linked.
- No subdivision of $K_{a,t}$ $\Rightarrow$ subwall W' with no attaching parts, all vertices $< a$ neighbors in A, $|V(W')| \geq M$.
- Local vertices of vortices cut off by small cuts $\rightarrow G'$.
- Contract vortex interiors $\Rightarrow \leq M$ vertices of degree < 6.

$$6(|V(G')| - M) + (19a - 6)M \leq 2|E(G')| \leq 6|V(G')| + 6g \; ↯$$