There exist graphs of minimum degree $\Omega(a\sqrt{\log a})$ that do not contain K_a as a minor. However, it turns out that assuming sufficiently large connectivity, such graphs must have bounded number of vertices. Indeed, the following claim holds.

Theorem 1 (Norin and Thomas)**.** *For every a there exists N such that every a-connected graph G with at least N vertices either*

- *contains K_a as a minor, or*
- *is obtained from a planar graph by adding at most $a-5$ apex vertices.*

The proof of this theorem is extremely involved. Instead, we will show a much simpler claim due to Böhme, Kawarabayashi, Maharry and Mohar, that nevertheless showcases some of the ideas of the proof.

Theorem 2. *For all a, k, s, t, there exists N such that every $(3a+2)$-connected graph of minimum degree at least $20a$ and with at least N vertices either*

- *contains $sK_{a,k}$ (s disjoint copies of $K_{a,k}$) as a minor, or*
- *contains a subdivision of $K_{a,t}$.*

Relating this to Theorem 1, note that K_a is a minor of $K_{a-1,a}$ (contract a perfect matching). In particular, we have the following (for a replaced by $a-1$, $k=t=a$, and $s=1$):

Corollary 3. *For every a there exists N such that every $(3a-1)$-connected graph of minimum degree at least $20a$ and with at least N vertices contains K_a as a minor.*

Moreover, for graphs of bounded maximum degree, the second outcome of Theorem 2 does not occur, and we have the following.

Corollary 4. *For all a, k, s, t, there exists N such that every $(3a+2)$-connected graph of minimum degree at least $20a$, maximum degree less than t, and with at least N vertices contains $sK_{a,k}$ as a minor.*

1 Within path decompositions

A graph M is *k-linked* if for any sequence $v_1, \ldots, v_{2k}$ vertices of M, there exists disjoint paths from v_1 to v_2, from v_3 to v_4, $\ldots$, from v_{2k-1} to v_{2k}. We need the following result.

Theorem 5. *A graph of average degree at least* $13k$ *contains a* k*-linked subgraph.*

Consider a path decomposition (Q, β) of a graph H, where $Q = x_0x_1 \ldots x_m$. For $i = 1, \ldots, m$, let $S_i = \beta(x_{i-1}) \cap \beta(x_i)$. We say that the decomposition is q-*linked* if $|S_1| = |S_2| = \ldots = |S_m| = q$ and H contains q vertex-disjoint paths $P_1, \ldots, P_q$ from S_1 to S_m. A vertex of $S_1 \cup \ldots S_m$ is an *interface* vertex, all other vertices are *internal*; note that each internal vertex belongs to exactly one bag. A *focus* F is a set of internal vertices, each belonging to a distinct bag different from $\beta(x_0)$ and $\beta(x_m)$; for $v \in F$, let i_v denote the unique vertex of Q such that $v \in \beta(x_{i_v})$, let $\beta_v = \beta(x_{i_v})$, $L_v = S_{i_v-1}$, and $R_v = S_{i_v}$.

A path P_i is F-*universal* if there exists a vertex w such that $V(P_i) \cap \beta_v = \{w\}$ for every $v \in F$, and F-*transversal* if $V(P_i) \cap \beta_v$ and $V(P_i) \cap \beta_{v'}$ are disjoint for all distinct $v, v' \in F$. We say that the paths $P_1, \ldots, P_q$ are F-*uniform* if each of them is F-universal or F-transversal.

Observation 6. *If* $s' \gg s, q$ *and* F' *is a focus of size at least* s'*, then there exists a focus* $F \subseteq F'$ *of size at least* s *such that each of the paths* $P_1, \ldots, P_q$ *are* F*-uniform.*

Proof. Process the paths $P_1, \ldots, P_q$ one by one. For each i, if there exists $w \in V(P_i)$ such that $\beta_v \cap V(P_i) = \{w\}$ for many (say b) vertices $v \in V(F')$, restrict F" to such vertices v, so that P_i is F'-universal. Otherwise, take every $(b + 2)$-nd vertex from F' in order along w; this ensures P_i is F'-transversal. □

We say that paths P_i and P_j are F-*adjacent* if for each $v \in F$, there exists a path in $H[\beta_v]$ from P_i to P_j disjoint from all other paths $P_1, \ldots, P_q$, and and F-*nonadjacent* if no such path exists for every $v \in F$. We say that the focus is *adjacency-uniform* if for all $i \neq j$, the paths P_i and P_j are either F-adjacent or F-nonadjacent. Similarly to the proof of Observation 6, we have the following.

Observation 7. *If* $s' \gg s, q$ *and* F' *is a focus of size at least* s'*, then there exists an adjacency-uniform focus* $F \subseteq F'$ *of size at least* s*.*

We say that the path decomposition is *internally* k*-connected with respect to* F if for each $v \in F$, there exists no separation (A, B) of $H[\beta_v]$ of order less than k such that $\{v\} \cup L_v \cup R_v \subseteq V(A)$ and $V(B) \not\subseteq V(A)$. It has *internally minimum degree at least* d *with respect to* F if for each $v \in F$, all vertices in $\beta_v \setminus (L_v \cup R_v)$ have degree at least d.

Lemma 8. *For all a, k, s, t, q, there exists N_0 as follows. Let (Q, β) be a q-linked path decomposition of a graph H, and let $P_1, \ldots, P_q$ be the linking paths. Let F be a focus such that the decomposition is internally $(3a+2)$-connected and internally has minimum degree at least $20a-4$ with respect to F. If $V(P_1) \cap \beta_v \subseteq \{v\} \cup L_v \cup R_v$ for each $v \in F$, then let $H' = H - E(P_1)$, otherwise let $H' = H$. If $|F| \geq N_0$, then either*

- *H contains $sK_{a,k}$ as a minor, or*
- *H' contains a subdivision of $K_{a,t}$.*

Proof. By Observations 6 and 7, we can assume F is adjacency-uniform and $P_1, \ldots, P_q$ are F-uniform. Without loss of generality, we can assume paths $P_1, \ldots, P_{c'}$ are F-transversal and the remaining ones are F-apex; let A' denote their set, and for $P \in A'$, let w_P be the vertex in which P intersects β_v for $v \in F$. Let Γ be the graph on paths $P_1, \ldots, P_{c'}$, where the two paths are adjacent iff they are F-adjacent. Let $\{P_1, \ldots, P_c\}$ be the component of $\Gamma - A'$ containing P_1, and let $A \subseteq A'$ consist of paths with a neighbor in this component.

Let l and r be the leftmost and the rightmost vertex of F in the path Q, and let $L = L_l$ and $R = R_r$. Let H_0 be the graph consisting of the segments of $P_1, \ldots, P_c$ between L and R and for each $v \in F$, the connected component of $H[\beta(v)] - A$ intersecting these segments. Note that H_0 is disjoint from $P_{c+1}, \ldots, P_q$. Let $B = \{w_P : P \in A\}$. Let H_1 be the subgraph of H obtained from H_0 by adding B and the edges from these vertices to H_0. Note that H_1 is separated by $L \cup B \cup R$ from the rest of H.

If there are many vertices $v \in F$ such that some $x_v \in \beta_v \cap V(H_0)$ has neighbors in at least $a+1$ of the paths $P_1, \ldots, P_c$, then excluding the path on which x_v lies and using the pigeonhole principle, we can assume many such vertices x_v have a neighbor on the same a of these paths and do not lie on them; contracting the appropriate path segments, we obtain a minor of $sK_{a,k}$ in H. Hence, by removing all v such that x_v exists from F, we can assume that for each $v \in F$, every vertex in $\beta_v \cap V(H_0)$ has neighbors in at most a of the paths $P_1, \ldots, P_c$, and in particular has at most $2a$ neighbors in $(L_v \cup R_v) \cap V(H_0)$.

If many vertices $v \in F$ have at least a neighbors in B, then we similarly obtain $K_{a,t} \subseteq H - E(P_1)$, and thus we can analogously assume each $v \in F$ has at most $a-1$ neighbors in B. Since the decomposition internally has minimum degree at least $20a-4 > 3a-1$ with respect to F, v has a neighbor $v' \in \beta_v \setminus (L_v \cup R_v \cup B)$.

If for many $v \in F$, there exist at least a disjoint paths in $H_1 - (\{v\} \cup L_v \cup R_v)$ from v' to B, then we similarly obtain a subdivision of $K_{a,t}$ in H' (using the assumption that $V(P_1) \cap \beta_v \subseteq \{v\} \cup L_v \cup R_v$ if $H' \neq H$). Hence, we can assume that this is not the case for any $v \in F$, and thus there exists a set X_v of at most $a-1$ vertices separating v' from B in $H_1 - (\{v\} \cup L_v \cup R_v)$. Let C_v be the component of $H_1 - (\{v\} \cup L_v \cup R_v \cup X_v)$ containing v'. Note that C_v has minimum degree at least $20a - 4 - 3a = 17a - 4$. By Theorem 5, there exists an $(a+1)$-linked subgraph $M_v \subseteq C_v$.

Since the decomposition is internally $(3a+2)$-connected with respect to F, H_1 contains $3a+2$ disjoint paths from M_v to $\{v\} \cup L_v \cup R_v$; by the previous paragraph, at least $2a+2$ from them end in $(L_v \cup R_v) \setminus B$. Consider such a system $\mathcal{L}_v$ of $2a+2$ paths with minimum number of edges outside $P_1 \cup \ldots \cup P_c$, If a path P_i intersects at least two paths from $\mathcal{L}_v$, then the minimality implies that one of the paths from $\mathcal{L}_v$ follows it to L_v and another one to R_v. If P_i is intersected only once, we can freely choose whether the path from $\mathcal{L}_v$ follows P_i to L_v or to R_v. Hence, we can balance the numbers and assume $\mathcal{L}_v$ contains $a+1$ paths to L_v and $a+1$ paths to R_v.

Moreover, consider any vertices $v_1, v_2 \in F$ such that at least a vertices of F appear between v_1 and v_2 on Q, and any subsets $X \subseteq L_{v_1} \cap V(H_0)$ and $Y \subseteq R_{v_2} \cap V(H_0)$ of size $a+1$. We claim the part of H_0 between L_{v_1} and R_{v_2} contains $a+1$ disjoint paths from X to Y. Indeed, deleting a vertices Z cannot separate X from Y: there exists $v \in F$ between v_1 and v_2 with β_v disjoint from Z, and a path P_i from X to β_v and $P_{i'}$ from Y to β_v disjoint from Z.

For $v \in F$ and $j = 1, \ldots, a+1$, let $\{y_{v,j}\} = L_v \cap P_j$. For sufficiently distant $u, v \in F$ and any $b \in \{2, \ldots, a+1\}$ we can obtain disjoint paths S_j from $u_{u,j}$ to $y_{v,j}$ and a disjoint path T from S_1 to S_b as follows: there exists an edge $P_{k_1}, P_{k_2} \in \Gamma$ for some $i, j \leq c$. Use the path systems from the previous two paragraps to connect $y_{u,1}$ and $y_{u,b}$ to y_{w,k_1} and y_{w,k_1} for some w between u and v, take T in $H_0 \cap \beta_w$, then again use the path systems to match the ends to $y_{v,j}$.

Using these jumps and contracting the appropriate segments of S_1, we obtain a minor of $sK_{a,k}$ in H. □

2 Within tree decompositions

A tree decomposition (T, β) of a graph G is *linked* if for any $x, y \in V(T)$ and an integer k, either G contains k vertex-disjoint paths from $\beta(x)$ to $\beta(y)$, or there exists $z \in V(T)$ separating x from y in T such that $|\beta(z)| < k$. A tree decomposition is *nondegenerate* if no two bags are the same.

Theorem 9 (Thomas)**.** *Every graph G has a nondegenerate linked tree decomposition of width $tw(G)$.*

We can now prove Theorem 2 for graphs of bounded treewidth.

Lemma 10. *For all a, k, s, t, ω, there exists N such that every $(3a + 2)$-connected graph G of minimum degree at least $20a$, treewidth at most ω, and with at least N vertices either contains $sK_{a,k}$ as a minor, or contains a subdivision of $K_{a,t}$.*

Proof. Let (T, β) be an optimal non-degenerate linked tree decomposition of F. If T contains a long path, find a long segment of this path such that all bags on it have size at least q and many have size exactly q. Contracting along the path, we obtain a q-linked path decomposition. Otherwise, T has a vertex of large degree. Contracting subtrees and adding the root bag to all bags, we obtain a (trivially) linked path decomposition. Choose internal vertices in its bags arbitrarily to obtain a focus and apply Lemma 8. □

3 Using the structure theorem

As we have seen in the homework assignment, in the local version of the structure theorem with respect to a prescribed wall W, we can assume:

- Up to 3-separations, W is drawn planarly in the surface part of the decomposition.
- Each vortex F with boundary $v_0v_1 \ldots v_m$ has a path decomposition $(v_1 \ldots v_m, \beta)$ such that
  - $\beta(v_i) \cap \{v_0, \ldots, v_m\} = \{v_{i-1}, v_i\}$, and
  - considered as a decomposition of $P + v_0v_1 \ldots v_m$, it is q-linked for some bounded q.

A boundary vertex of a vortex F is *local* if all but at most four neighbors of its neighbors belong to the vortex or are the apex vertices. The vortex F is *N-wide* if there exist indices $1 \le i_1 < i_2 < \ldots < i_N \le m - 1$ such that vertices v_{i_j} for $j = 1, \ldots, N$ are local and there exists a path P and paths $Z_1, \ldots, Z_N$ from $v_{i_1}, \ldots, v_{i_N}$ to P whose ends in P are in order, such that $P \cup Z_1 \cup \ldots \cup Z_N$ is disjoint from F and the apex vertices except for $\{v_{i_1}, \ldots, v_{i_N}\}$.

Lemma 11. *If the decomposition of a $(3a+2)$-connected graph G of minimum degree at least $20a$ contains a sufficiently wide vortex, then G either contains $sK_{a,k}$ as a minor, or contains a subdivision of $K_{a,t}$.*

Proof. Add the apex vertices to the vortex. Contract the paths $Z_1, \ldots, Z_N$ and appropriate subpaths of P to obtain a path with vertex set $v_{i_1}, \ldots, v_{i_N}$. Modify the decomposition of the vortex plus this path: Join bags around these vertices to obtain a focus, merge the bags between them. Apply Lemma 8. □

In the proof of Theorem 2, we can assume $sK_{a,k}$ is not a minor of G, and thus the structure theorem applies. In view of Lemmas 10 and 11, it suffices to deal with the case G contains a large wall W and the corresponding decomposition does not contain a wide vortex. If many vertices of the embedded part have at least a neighbors among the apex vertices, we obtain $K_{a,t} \subseteq G$. Similarly, suppose many parts attach to cliques of size at most three in the embedded part; since G is $(3a+2)$-connected, in each such part we have more than a disjoint paths from a vertex to the apices, obtaining a subdivision of $K_{a,t}$ in G. Hence, most of the embedded part is indeed a subgraph of G; and since G has minimum degree at least $20a$, most of the embedded part has minimum degree more than $19a$.

If W cannot be separated by a small cut from many of the local vertices of one of the vortices, then there exist many paths from these vertices to the outer cycle of W, and (using Erdős-Szekerés to ensure the right ordering of the ends), we conclude the vortex is wide. Otherwise, local vertices of vortices can be cut off by a number of vertices Y which is negligible compared to the size of W. Consider the Y-bridge of the embedded part containing W. After replacing each vortex by a vertex, remaining non-local boundary vertices (not in Y) have degree at least six, while almost all other vertices have degree more than $19a \gg 6$. Since the number of vertices of W is large compared to the number of exceptional vertices (of degree less than 6), this implies the average degree is too large (compared to the bound from the Euler's formula), a contradiction.