Minors

Definition

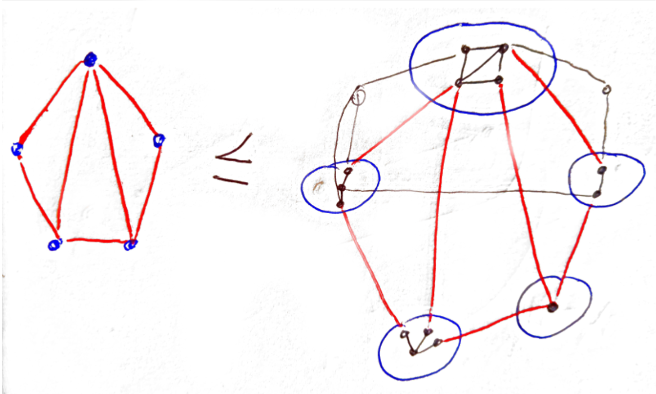
H is a minor of G if H is obtained from a subgraph of G by contracting vertex-disjoint connected subgraphs.

We write $H \preceq G$.

Definition

Model μ of a minor of H in G is a function s.t.

- $\mu(v_1), \ldots, \mu(v_k)$ (where $V(H) = \{v_1, \ldots, v_k\}$ are vertex-disjoint connected subgraphs of G, and
- for $e = uv \in E(H)$, $\mu(e)$ is an edge of G with one end in $\mu(u)$ and the other in $\mu(v)$.



Tree decompositions

Definition

A tree decomposition of a graph G is a pair (T, β), where

- T is a tree and $\beta(x) \subseteq V(G)$ for every $x \in V(T)$,
- for every $uv \in E(G)$, there exists $x \in V(T)$ s.t. $u, v \in \beta(x)$, and
- for every $v \in V(G)$, $\{x \in V(T) : v \in \beta(x)\}$ induces a non-empty connected subtree of T.

The width of the decomposition is $\max\{|\beta(x)| : x \in V(T)\} - 1$.

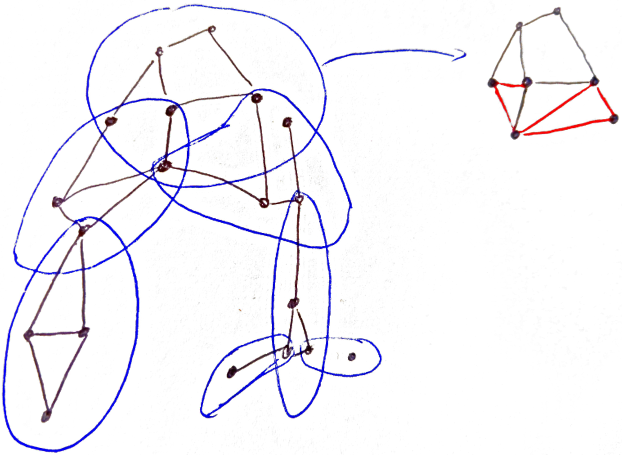
Treewidth $\mathrm{tw}(G)$: the min. width of a tree decomposition of G.

Lemma

$H \preceq G \Rightarrow tw(H) \leq tw(G)$.

Definition

Let (T, β) be a tree decomposition of G. The torso of $x \in V(T)$ is obtained from $G[\beta(x)]$ by adding cliques on $\beta(x) \cap \beta(y)$ for all $xy \in E(T)$.



Structural theorems

Theorem (Kuratowski)

$$K_5, K_{3,3} \not\preceq G \Leftrightarrow G \text{ is planar.}$$

Theorem (Robertson and Seymour)

For every planar graph H, there exists a constant c_H s.t.

$$H \not\preceq G \Rightarrow tw(G) \leq c_H.$$

Theorem (Wagner)

If $K_5 \not\preceq G$, then G has a tree decomposition in which each torso is either planar, or has at most 8 vertices.

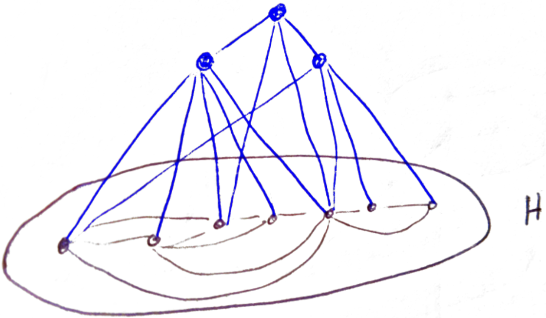
Apex vertices

Observation

If $K_n \not\preceq G - v$*, then* $K_{n+1} \not\preceq G$*.*

Definition

G is obtained from H by adding a apices if $H = G - A$ for some set $A \subseteq V(G)$ of size a.



Apex vertices in structural theorems

Observation

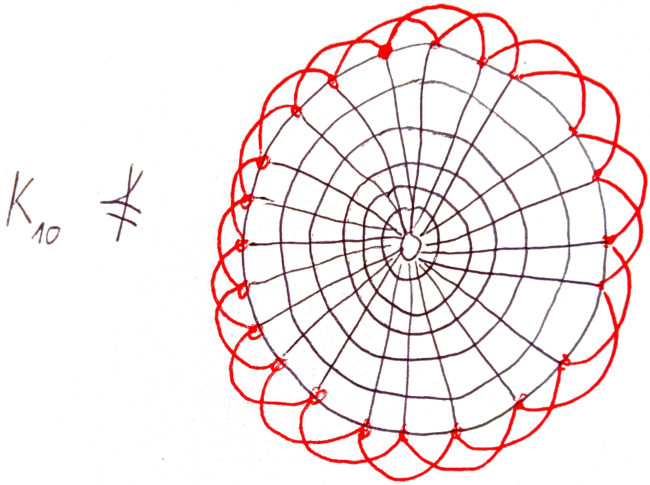
$K_6 \not\preceq$ planar + one apex.

Theorem (Robertson and Seymour)

For some fixed a, If $K_6 \not\preceq G$, then G has a tree decomposition in which each torso is either

- *obtained from a planar graph by adding at most a apices, or*
- *has at most a vertices.*

Vortices

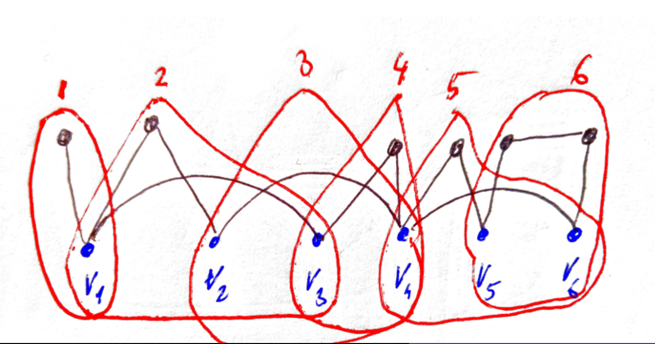


Vortices

Definition

A graph H is a vortex of depth d and boundary sequence $v_1, \ldots, v_k$ if H has a path decomposition (T, β) of width at most d such that

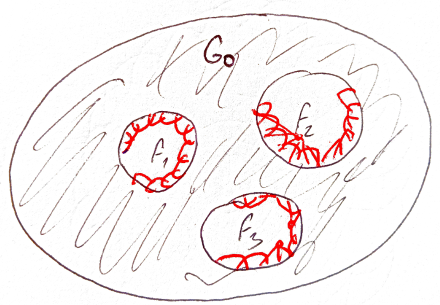
- $T = v_1 v_2 \ldots v_k$, and
- $v_i \in \beta(v_i)$ for $i = 1, \ldots, k$



Definition

For G_0 drawn in a surface, a graph G is an outgrowth of G_0 by m vortices of depth d if

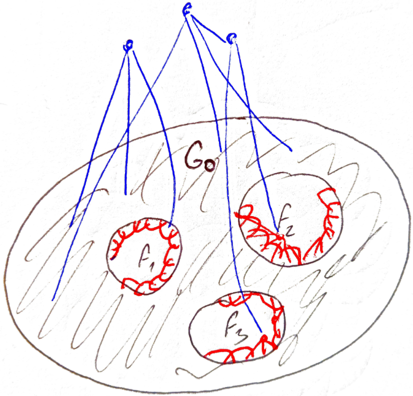
- $G = G_0 \cup H_1 \cup H_m$, where $H_i \cap H_j = \emptyset$ for distinct i and j,
- for all i, H_i is a vortex of depth d intersecting G only in its boundary sequence,
- for some disjoint faces $f_1, \ldots, f_k$ of G_0, the boundary sequence of H_i appears in order on the boundary of f_i.



Near-embeddability

Definition

A graph G is a-near-embeddable in a surface Σ if for some graph G_0 drawn in Σ, G is obtained from an outgrowth of G_0 by at most a vortices of depth a by adding at most a apices.



The structure theorem

Theorem (Robertson and Seymour)

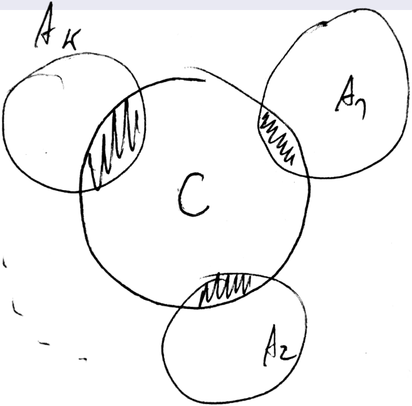
For every graph H, there exists a such that the following holds. If $H \not\preceq G$, then G has a tree decomposition such that each torso either

- *has at most a vertices, or*
- *is a-near-embeddable in some surface Σ in which <u>H cannot be drawn</u>.*

Definition

A location in G is a set of separations $\mathcal{L}$ such that for distinct $(A_1, B_1), (A_2, B_2) \in \mathcal{L}$, we have $A_1 \subseteq B_2$.

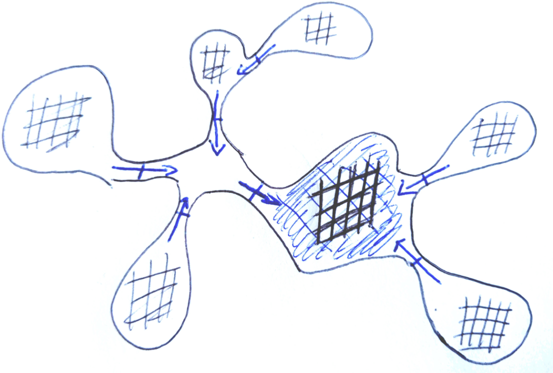
The center of the location is the graph C obtained from $\bigcap_{(A,B)\in\mathcal{L}} B$ by adding all edges of cliques with vertex sets $V(A \cap B)$ for $(A, B) \in \mathcal{L}$.



Local structure theorem

Theorem (Robertson and Seymour)

For every graph H, there exists a such that the following holds. If $H \not\preceq G$ and $\mathcal{T}$ is a tangle in G of order at least a, then there exists a location $\mathcal{L} \subseteq \mathcal{T}$ whose center is a-near-embeddable in some surface Σ in which H cannot be drawn.



From Local to Global

Generalization:

Theorem (Robertson and Seymour)

For every graph H, there exists a such that the following holds. If $H \not\preceq G$ and $W \subseteq V(G)$ has at most $3a$ vertices, then G has a tree decomposition (T, β) with root r s.t. each torso either

- *has at most $4a$ vertices, or*
- *is $4a$-near-embeddable in some surface Σ in which H cannot be drawn,*

and furthermore, $W \subseteq \beta(r)$ and the above holds for the torso of r + a clique on W.

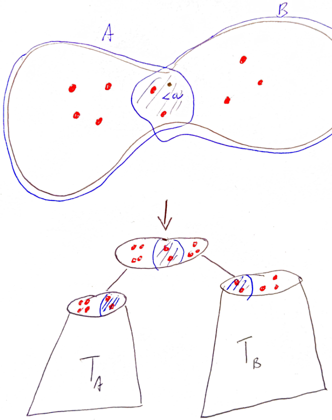
Case (a): W breakable

Separation (A, B) of order $< a$ such that $|W \setminus V(A)| \leq 2a$ and $|W \setminus V(B)| \leq 2a$:

Induction on

- A with $W_A = (W \setminus V(B)) \cup V(A \cap B)$ and
- B with $W_B = (W \setminus V(A)) \cup V(A \cap B)$.

Root bag with $\beta(r) = W \cup V(A \cap B)$.



Case (b): W not breakable

For every separation (A, B) of order $< a$, either $|W \setminus V(A)| > 2a$ or $|W \setminus V(B)| > 2a$:

$\mathcal{T} = \{(A, B) :$ separation of order $< a$, $|W \setminus V(A)| > 2a\}$ is a tangle of order a.

- Local Structure Theorem: location $\mathcal{L} \subseteq \mathcal{T}$ with a-near-embeddable center C.
- For $(A, B) \in \mathcal{L}$, induction on A with $W_A = (W \setminus V(B)) \cup V(A \cap B)$.
- Root bag with $\beta(r) = V(C) \cup W$: at most $3a$ apices.