Reminder

Let **V** be a vector space over a field **F**. Let $B = v_1, \ldots, v_n$ be a basis of **V**.

Definition

The coordinates of a vector $v \in \mathbf{V}$ with respect to the basis B are given by the (unique) vector $[v]_B = (\alpha_1, \ldots, \alpha_n) \in \mathbf{F}^n$ such that

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

Linearity of coordinates

Observation

Let $\mathbf{V}$ *be a vector space over field* $\mathbf{F}$*, and let* B *be a basis of* $\mathbf{V}$*.*

- *For every* $u, v \in \mathbf{V}$*,*

$$[u + v]_B = [u]_B + [v]_B.$$

- *For every* $v \in \mathbf{V}$ *and* $\alpha \in \mathbf{F}$*,*

$$[\alpha v]_B = \alpha[v]_B.$$

Instead of computing in (possibly complicated) $\mathbf{V}$, compute in $\mathbf{F}^{\dim(\mathbf{V})}$!

Example

Consider the following bases of $\mathbf{R}^2$:

- $B_1 = (1,0),(0,1)$
- $B_2 = (1,1),(-1,1)$
- $B_3 = (1,2),(3,4)$

Let $v = (3,2)$. Then

- $[v]_{B_1} = (3,2)$, since $(3,2) = 3(1,0) + 2(0,1)$
- $[v]_{B_2} = (5/2,-1/2)$, since $(3,2) = \frac{5}{2}(1,1) - \frac{1}{2}(-1,1)$
- $[v]_{B_3} = (-3,2)$, since $(3,2) = -3(1,2) + 2(3,4)$

Coordinate transformation

- Let $B = b_1, \ldots, b_n$ and $C = c_1, \ldots, c_n$ be two bases of a vector space **V**.
- let $[v]_B = (\beta_1, \ldots, \beta_n)$ and $[v]_C = (\gamma_1, \ldots, \gamma_n)$.

What is the relationship between $[v]_B$ and $[v]_C$?

- For $i = 1, \ldots, n$, let $[b_i]_C = (\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{n,i})$.

$$\begin{aligned} v &= \beta_1 b_1 + \ldots + \beta_n b_n \\ &= \beta_1 \sum_{j=1}^{n} \alpha_{j,1} c_j + \ldots + \beta_n \sum_{j=1}^{n} \alpha_{j,n} c_j \\ &= \gamma_1 c_1 + \ldots + \gamma_n c_n. \end{aligned}$$

Hence, for $j = 1, \ldots, n$:

$$\gamma_j = \alpha_{j,1}\beta_1 + \alpha_{j,2}\beta_2 + \ldots + \alpha_{j,n}\beta_n$$

Basis transition matrix

Definition

Let $B = b_1, \ldots, b_n$ and C be two bases of a vector space $\mathbf{V}$. The matrix

$$[\mathrm{id}]_{B,C} = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \cdots & \alpha_{2,n} \\ & \cdots & \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{pmatrix},$$

where $[b_i]_C = (\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{n,i})$ for $i = 1, \ldots, n$, is the basis transition matrix from B to C.

The i-th column of $[\mathrm{id}]_{B,C}$ gives the coordinates of b_i in C.

Lemma

For any vector v,

$$[v]_C^T = [id]_{B,C}[v]_B^T.$$

Properties of basis transitions

Lemma

Let $B = b_1, \dots, b_n$, C, and D be bases of a vector space $\mathbf{V}$.

$$[id]_{B,C} = [id]_{C,B}^{-1}$$
$$[id]_{B,D} = [id]_{C,D}[id]_{B,C}$$

Proof.

The i-th column of $[\mathrm{id}]_{C,B}[\mathrm{id}]_{B,C}$ is

$$[\mathrm{id}]_{C,B}[\mathrm{id}]_{B,C}e_i^T = [\mathrm{id}]_{C,B}[\mathrm{id}]_{B,C}[b_i]_B^T = [\mathrm{id}]_{C,B}[b_i]_C^T = [b_i]_B^T = e_i^T,$$

hence $[\mathrm{id}]_{C,B}[\mathrm{id}]_{B,C} = I$.

$\square$

Properties of basis transitions

Lemma

Let $B = b_1, \dots, b_n$, C, and D be bases of a vector space $\mathbf{V}$.

$$[id]_{B,C} = [id]_{C,B}^{-1}$$
$$[id]_{B,D} = [id]_{C,D}[id]_{B,C}$$

Proof.

The i-th column of $[\mathrm{id}]_{C,D}[\mathrm{id}]_{B,C}$ is

$$[\mathrm{id}]_{C,D}[\mathrm{id}]_{B,C}e_i^T = [\mathrm{id}]_{C,D}[\mathrm{id}]_{B,C}[b_i]_B^T = [\mathrm{id}]_{C,D}[b_i]_C^T = [b_i]_D^T,$$

the same as the i-th column of $[\mathrm{id}]_{B,D}$.

□

Computing a basis transition matrix

Problem

Let $B = (1,1), (-1,1)$ and $C = (1,2), (3,4)$. Compute the basis transition matrix $[id]_{B,C}$.

Let $D = (1,0), (0,1)$ be the standard basis. Then

$$[\mathrm{id}]_{B,D} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad [\mathrm{id}]_{C,D} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

- $[\mathrm{id}]_{B,C} = [\mathrm{id}]_{D,C}[\mathrm{id}]_{B,D} = [\mathrm{id}]_{C,D}^{-1}[\mathrm{id}]_{B,D}$.
- Recall: if X is regular, then $\mathrm{RREF}(X|Y) = (I|X^{-1}Y)$.

$$\mathrm{RREF}\left(\begin{array}{cc|cc} 1 & 3 & 1 & -1 \\ 2 & 4 & 1 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & 7/2 \\ 0 & 1 & 1/2 & -3/2 \end{array}\right)$$

$$[\mathrm{id}]_{B,C} = \begin{pmatrix} -1/2 & 7/2 \\ 1/2 & -3/2 \end{pmatrix}$$

Using a basis transition matrix

Problem

Let $B = (1,1), (-1,1)$ and $C = (1,2), (3,4)$. If $[v]_B = (5/2, -1/2)$, what are the coordinates of v with respect to C?

$$[v]_C^T = [\mathrm{id}]_{B,C}[v]_B^T = \begin{pmatrix} -1/2 & 7/2 \\ 1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 5/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix},$$

hence

$$[v]_C = (-3, 2).$$

Using a basis transition matrix

Problem

Let $B = (1,1), (-1,1)$ and $C = (1,2), (3,4)$. If $[v]_B = (5/2, -1/2)$, what are the coordinates of v with respect to C?

$$[v]_C^T = [\mathrm{id}]_{B,C}[v]_B^T = \begin{pmatrix} -1/2 & 7/2 \\ 1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 5/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix},$$

hence

$$[v]_C = (-3, 2).$$

Note: only practical when transforming several vectors. Otherwise, compute $v = \frac{5}{2}(1,1) - \frac{1}{2}(-1,1) = (3,2)$ and determine $[v]_C$ by solving linear equations.

Application: (idea of) fast polynomial multiplication

Let p, q be polynomials of degree at most n, let $\alpha_1, \ldots, \alpha_{2n+1}$ be distinct complex numbers.

- The straightforward algorithm to compute pq needs $\approx n^2$ operations.
- Given $p(\alpha_1), \ldots, p(\alpha_{2n+1}), q(\alpha_1), \ldots, q(\alpha_{2n+1})$:
  - The values

$$\begin{aligned}(pq)(\alpha_1) &= p(\alpha_1)q(\alpha_1)\\ (pq)(\alpha_2) &= p(\alpha_2)q(\alpha_2)\\ &\cdots\\ (pq)(\alpha_{2n+1}) &= p(\alpha_{2n+1})q(\alpha_{2n+1})\end{aligned}$$

    can be computed using $\approx n$ operations.
  - These values uniquely determine $pq \in \mathcal{P}_{2n}$.

Application: (idea of) fast polynomial multiplication

Let $B = 1, x, x^2, \ldots, x^{2n}$, $C = \ell_1, \ldots, \ell_{2n+1}$ be bases of $\mathcal{P}_{2n}$, where

- $\ell_i(\alpha_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

For a polynomial $p = \pi_0 + \pi_1 x + \ldots + \pi_{2n} x^{2n}$:

- $[p]_B = (\pi_0, \pi_1, \ldots, \pi_{2n})$
- $[p]_C = (p(\alpha_1), \ldots, p(\alpha_{2n+1}))$

Application: (idea of) fast polynomial multiplication

To compute the coefficients $[pq]_B$ of pq:

- Compute

$$[p]_C^T = [\mathrm{id}_{B,C}][p]_B^T$$
$$[q]_C^T = [\mathrm{id}_{B,C}][q]_B^T$$

- Compute $[pq]_C$ by multiplying $[p]_C$ and $[q]_C$ element-by-element.
- Compute $[pq]_B^T = [\mathrm{id}_{C,B}][pq]_C^T$.

Application: (idea of) fast polynomial multiplication

To compute the coefficients $[pq]_B$ of pq:

- Compute

$$[p]_C^T = [\mathrm{id}_{B,C}][p]_B^T$$
$$[q]_C^T = [\mathrm{id}_{B,C}][q]_B^T$$

- Compute $[pq]_C$ by multiplying $[p]_C$ and $[q]_C$ element-by-element.
- Compute $[pq]_B^T = [\mathrm{id}_{C,B}][pq]_C^T$.

To perform the multiplications by $[\mathrm{id}_{B,C}]$ and $[\mathrm{id}_{C,B}]$ efficiently:

- Choose $\alpha_1, \dots, \alpha_{2n+1}$ cleverly
  - so that $[\mathrm{id}_{B,C}]$ and $[\mathrm{id}_{C,B}]$ have very special form
- FFT algorithm

Needs only $\approx n \log n$ operations.

Linear functions

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.

Definition

A function $f : \mathbf{U} \to \mathbf{V}$ is linear if

- For every $u_1, u_2 \in \mathbf{U}$,

$$f(u_1 + u_2) = f(u_1) + f(u_2).$$

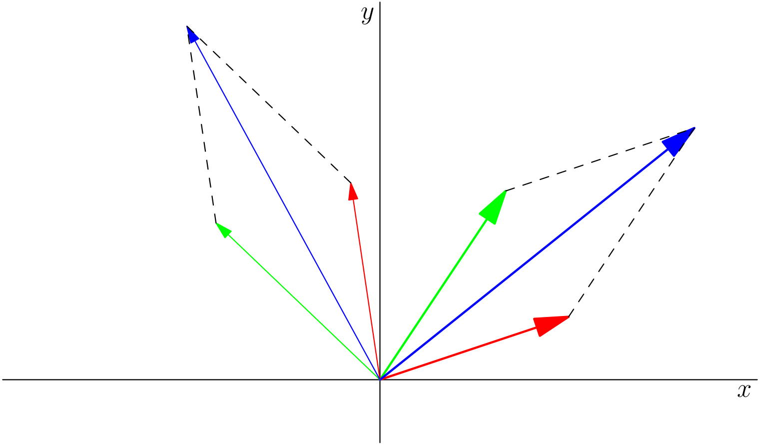
- For every $u \in \mathbf{U}$ and $\alpha \in \mathbf{F}$,

$$f(\alpha u) = \alpha f(u).$$

Also called linear maps, transformations, operators, ...

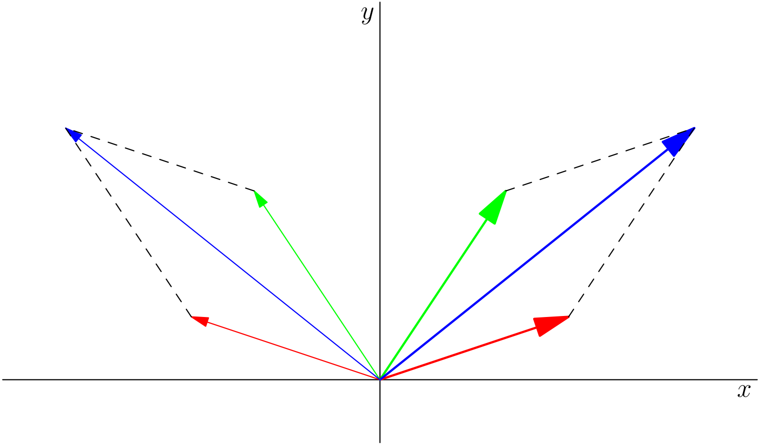
Examples of linear functions

- Mapping of v to $[v]_B$.
- $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = 2x + 3y$.
- For any $m \times n$ matrix A, $f : \mathbf{R}^{n\times 1} \to \mathbf{R}^{m\times 1}$ defined by $f(x) = Ax$.
- Let $\mathbf{S}$ be the vector space of infinite sequences. "shift left" $D : \mathbf{S} \to \mathbf{S}$, $D(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$
- Derivative is a linear function from $\mathcal{P}^n$ to $\mathcal{P}^{n-1}$.
- $g : \mathbf{U} \to \mathbf{V}$, $g(u) = o$.
- $\mathrm{id} : \mathbf{V} \to \mathbf{V}$, $\mathrm{id}(v) = v$.



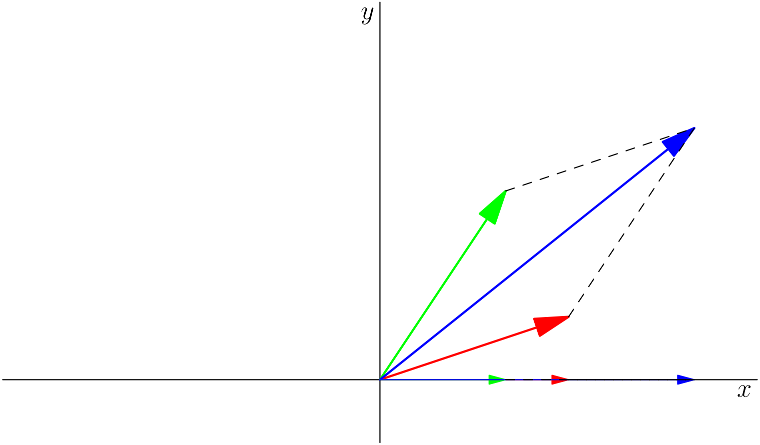
Rotation by 80 degrees.

Linear transformations of the plane



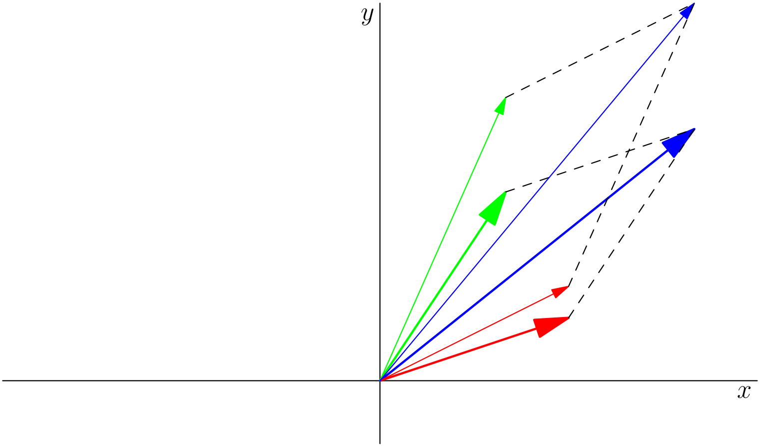
Reflection across the *y* axis.

Linear transformations of the plane



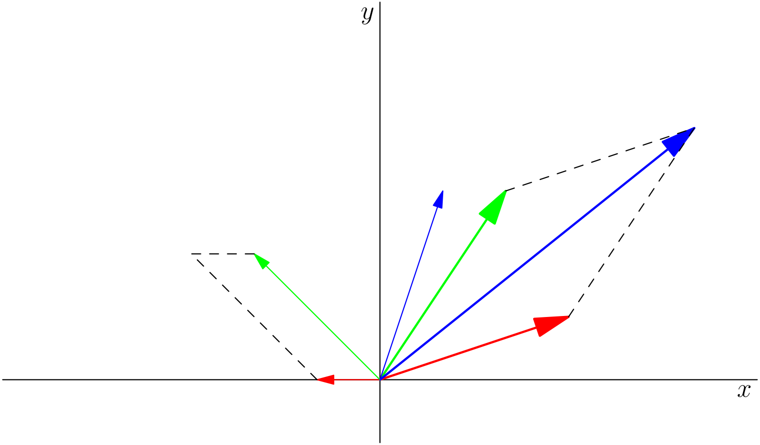
Projection to the *x* axis.

Linear transformations of the plane



Enlarging by half in the *y* direction.

Linear transformations of the plane



Translation by $(-4, -1)$ is not linear.

Properties of linear functions

Lemma

If $f : \mathbf{U} \to \mathbf{V}$ is linear, then

- $f(o) = o$
- $f(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1 f(u_1) + \ldots + \alpha_n f(u_n)$.

Linear functions and bases

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$, let $B = u_1, \ldots, u_n$ be a basis of $\mathbf{U}$.

Lemma

For every $v_1, \ldots, v_n \in \mathbf{V}$, there exists a unique linear function $f : \mathbf{U} \to \mathbf{V}$ such that

$$f(u_1) = v_1, \ldots, f(u_n) = v_n.$$

Proof.

For every $u = \alpha_1 u_1 + \ldots + \alpha_n u_n \in \mathbf{U}$, let

$$f(u) = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

- Linear by the linearity of coordinates.
- $f(u_i) = v_i$ for $i = 1, \ldots, n$.

$\square$

Linear functions and bases

Let **U** and **V** be vector spaces over the same field **F**, let $B = u_1, \ldots, u_n$ be a basis of **U**.

Lemma

For every $v_1, \ldots, v_n \in \mathbf{V}$, there exists a unique linear function $f : \mathbf{U} \to \mathbf{V}$ such that

$$f(u_1) = v_1, \ldots, f(u_n) = v_n.$$

Proof.

Uniqueness:

$$f(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1 f(u_1) + \ldots + \alpha_n f(u_n) = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

by linearity.

$\square$

Matrix of a linear function

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$,

- let $B = u_1, \ldots, u_n$ be a basis of $\mathbf{U}$,
- let C be a basis of $\mathbf{V}$.

Definition

For a linear function $f : \mathbf{U} \to \mathbf{V}$, the matrix of the function with respect to bases B and C is the $(\dim \mathbf{V} \times \dim \mathbf{U})$-matrix whose i-th column consists of the coordinates of $f(u_i)$:

$$[f]_{B,C} = \left([f(u_1)]_C^T \mid [f(u_2)]_C^T \mid \ldots \mid [f(u_n)]_C^T\right).$$

- $[f]_{B,C}$ uniquely determines f, and
- for any $(\dim \mathbf{V} \times \dim \mathbf{U})$-matrix A, there exists a linear function $f : \mathbf{U} \to \mathbf{V}$ such that $[f]_{B,C} = A$.

Example(1)

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection across the y axis,

$$f(x, y) = (-x, y).$$

Let $B = C = (1, 0), (0, 1)$ be the standard basis. Then

$$f(1, 0) = (-1, 0)$$
$$f(0, 1) = (0, 1)$$
$$[f]_{B,C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example(2)

Let $d : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative. Let

- $B = 1, x, x^2, x^3$ a basis of $\mathcal{P}^3$,
- $C_1 = 1, x, x^2$ a basis of $\mathcal{P}^2$,
- $C_2 = 1, 1 + x, 1 + x + x^2$ another basis of $\mathcal{P}^2$.

$$
\begin{aligned}
d(1) &= 0 \\
d(x) &= 1 \\
d(x^2) &= 2x = 2(1 + x) - 2 \\
d(x^3) &= 3x^2 = 3(1 + x + x^2) - 3(1 + x) \\
[d]_{B,C_1} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\
[d]_{B,C_2} &= \begin{pmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}
\end{aligned}
$$

Meaning of the matrix of a linear function

Let **U** and **V** be vector spaces over the same field **F**,

- let $B = u_1, \ldots, u_n$ be a basis of **U**,
- let C be a basis of **V**.

Lemma

If $f : \mathbf{U} \to \mathbf{V}$ is a linear function and $u \in \mathbf{U}$, then

$$[f(u)]_C^T = [f]_{B,C}[u]_B^T.$$

Instead of computing the function directly, we can evaluate it on coordinates using matrix multiplication.

Meaning of the matrix of a linear function

Let **U** and **V** be vector spaces over the same field **F**,

- let $B = u_1, \ldots, u_n$ be a basis of **U**,
- let C be a basis of **V**.

Lemma

If $f : \mathbf{U} \to \mathbf{V}$ is a linear function and $u \in \mathbf{U}$, then

$$[f(u)]_C^T = [f]_{B,C}[u]_B^T.$$

Proof.

$[f]_{B,C}[u_i]_B^T = [f]_{B,C}e_i^T$ is the i-th column of $[f]_{B,C}$, equals $[f(u_i)]_C^T$.
If $[u]_B = (\alpha_1, \ldots, \alpha_n)$, then

$$[f]_{B,C}[u]_B^T = \sum_{i=1}^{n} \alpha_i [f]_{B,C}[u_i]_B^T = \sum_{i=1}^{n} \alpha_i [f(u_i)]_C^T = [f(u)]_C^T.$$

Composition

Let **U**, **V**, and **W** be vector spaces over the same field **F**, with bases $B = u_1, \ldots, u_n$, C, and D, respectively.

Lemma

For any linear $f : \mathbf{U} \to \mathbf{V}$ *and* $g : \mathbf{V} \to \mathbf{W}$*,*

$$[gf]_{B,D} = [g]_{C,D}[f]_{B,C}.$$

Proof.

The i-th column of $[g]_{C,D}[f]_{B,C}$ is

$$[g]_{C,D}[f]_{B,C}e_i^T = [g]_{C,D}[f]_{B,C}[u_i]_B^T = [g]_{C,D}[f(u_i)]_C^T = [g(f(u_i))]_D^T,$$

which is the same as the i-th column of $[gf]_{B,D}$. $\square$

Basis transition matrix vs. linear functions

- Basis transition matrix $[\mathrm{id}]_{B,C}$ maps coordinates of v with respect to B to coordinates of v with respect to C.
- I.e., it is the matrix of the identity function id with respect to bases B and C.
- Hence the notation $[\mathrm{id}]_{B,C}$.

Isomorphisms

Definition

A linear function $f : \mathbf{U} \to \mathbf{V}$ is an isomorphism if f is bijective (1-to-1 and onto).

- If there exists an isomorphism from **U** to **V**, then **U** and **V** are isomorphic.
- f "renames" the elements of **U** to elements of **V**, preserving their linear combinations.
  - In particular, $\dim(\mathbf{U}) = \dim(\mathbf{V})$.
- Since f is bijective, it has an inverse f^{-1} defined by

$$f^{-1}(v) = u \text{ if and only if } f(u) = v.$$

  - $f(f^{-1}(v)) = v$ for every $v \in \mathbf{V}$
  - $f^{-1}(f(u)) = u$ for every $u \in \mathbf{U}$

Inverse

Let **U** and **V** be vector spaces over the same field **F**,

- let $B = u_1, \ldots, u_n$ be a basis of **U**,
- let C be a basis of **V**.

Lemma

If $f : \mathbf{U} \to \mathbf{V}$ is an isomorphism, then f^{-1} is linear and

$$[f^{-1}]_{C,B} = [f]^{-1}_{B,C}.$$

Proof.

Linearity: let $v_1, v_2 \in \mathbf{V}, \alpha \in \mathbf{F}$.

$$\begin{aligned} f^{-1}(v_1 + v_2) &= f^{-1}\big(f(f^{-1}(v_1)) + f(f^{-1}(v_2))\big) \\ &= f^{-1}\big(f(f^{-1}(v_1) + f^{-1}(v_2))\big) = f^{-1}(v_1) + f^{-1}(v_2) \\ f^{-1}(\alpha v_1) &= f^{-1}\big(\alpha f(f^{-1}(v_1))\big) = f^{-1}\big(f(\alpha f^{-1}(v_1))\big) = \alpha f^{-1}(v_1) \end{aligned}$$

$\square$

Inverse

Let **U** and **V** be vector spaces over the same field **F**,

- let $B = u_1, \ldots, u_n$ be a basis of **U**,
- let C be a basis of **V**.

Lemma

If $f : \mathbf{U} \to \mathbf{V}$ is an isomorphism, then f^{-1} is linear and

$$[f^{-1}]_{C,B} = [f]_{B,C}^{-1}.$$

Proof.

The i-th column of $[f^{-1}]_{C,B}[f]_{B,C}$ is

$$\begin{aligned}[f^{-1}]_{C,B}[f]_{B,C}e_i^T &= [f^{-1}]_{C,B}[f]_{B,C}[u_i]_B^T = [f^{-1}]_{C,B}[f(u_i)]_C^T \\ &= [f^{-1}(f(u_i))]_B^T = [u_i]_B^T = e_i^T,\end{aligned}$$

hence $[f^{-1}]_{C,B}[f]_{B,C} = I$. $\square$

Example: linear transformations of the plane

Problem

Let p be the line in $\mathbf{R}^2$ *through the origin in* 30 *degrees angle. To which point is* (x, y) *mapped by reflection across the p axis?*

The reflection across the p axis defines an isomorphism $g : \mathbf{R}^2 \to \mathbf{R}^2$.

Let r be the rotation by 30 degrees and f the reflection across the x axis. Then,

$$g = rfr^{-1},$$

and $[g] = [r][f][r]^{-1}$ with respect to the standard basis.

Example: linear transformations of the plane

Problem

Let p be the line in $\mathbf{R}^2$ through the origin in 30 degrees angle. To which point is (x, y) mapped by reflection across the p axis?

$$r(1,0) = (\sqrt{3}/2, 1/2) \qquad f(1,0) = (1,0)$$

$$r(0,1) = (-1/2, \sqrt{3}/2) \qquad f(0,1) = (0,-1)$$

$$[r] = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \qquad [f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[g] = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Hence, $g(x, y) = (x/2 + \sqrt{3}y/2, \sqrt{3}x/2 - y/2)$.

Example: composition of rotations

Let $r_\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation by angle α.

$$r_\alpha(1,0) = (\cos\alpha, \sin\alpha)$$
$$r_\alpha(0,1) = (-\sin\alpha, \cos\alpha)$$
$$[r_\alpha] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Note that $r_{\alpha+\beta} = r_\alpha r_\beta$, and $[r_{\alpha+\beta}] = [r_\alpha][r_\beta]$:

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

Therefore,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$