Elementary operation matrices: row addition

For $t \neq a$, let $A^{(n,t,a)}$ be the $n \times n$ matrix such that

$$A^{(n,t,a)}_{r,c} = \begin{cases} 1 & \text{if } r = c \text{, or if } r = t \text{ and } c = a \\ 0 & \text{otherwise} \end{cases}$$

$$A^{(n,t,a)} = I + e_t e_a^T$$

Example:

$$A^{(5,2,4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Elementary operation matrices: row addition

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$$A^{(n,t,a)} = I + e_t e_a^T$$

Lemma

If B is an $n \times m$ matrix, then $A^{(n,t,a)}B$ is obtained from B by the adding a-th row to the t-th row.

Elementary operation matrices: multiplying a row

For real number $\alpha \neq 0$, let $M^{(n,k,\alpha)}$ be the $n \times n$ matrix such that

$$M^{(n,k,\alpha)}_{r,c} = \begin{cases} 1 & \text{if } r = c \neq k \\ \alpha & \text{if } r = c = k \\ 0 & \text{otherwise} \end{cases}$$

$$M^{(n,k,\alpha)} = I + (\alpha - 1) e_k e_k^T$$

Example:

$$M^{(5,2,4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Elementary operation matrices: multiplying a row

For real number $\alpha \neq 0$, let $M^{(n,k,\alpha)}$ be the $n \times n$ matrix such that

$$M^{(n,k,\alpha)}_{r,c} = \begin{cases} 1 & \text{if } r = c \neq k \\ \alpha & \text{if } r = c = k \\ 0 & \text{otherwise} \end{cases}$$

$$M^{(n,k,\alpha)} = I + (\alpha - 1) e_k e_k^T$$

Lemma

If B is an $n \times m$ matrix, then $M^{(n,k,\alpha)} B$ is obtained from B by multiplying the k-th row by α.

Elementary operation matrices: exchanging rows

For $r_1 \neq r_2$, let $T^{(n,r_1,r_2)}$ be the $n \times n$ matrix such that

$$T^{(n,r_1,r_2)}_{r,c} = \begin{cases} 1 & \text{if } r = c \notin \{r_1, r_2\} \\ 1 & \text{if } r = r_1 \text{ and } c = r_2 \\ 1 & \text{if } r = r_2 \text{ and } c = r_1 \\ 0 & \text{otherwise} \end{cases}$$

Example:

$$T^{(5,2,4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Elementary operation matrices: exchanging rows

For $r_1 \neq r_2$, let $T^{(n,r_1,r_2)}$ be the $n \times n$ matrix such that

$$T^{(n,r_1,r_2)}_{r,c} = \begin{cases} 1 & \text{if } r = c \notin \{r_1, r_2\} \\ 1 & \text{if } r = r_1 \text{ and } c = r_2 \\ 1 & \text{if } r = r_2 \text{ and } c = r_1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma

If B is an $n \times m$ matrix, then $T^{(n,r_1,r_2)}B$ is obtained from B by exchanging the r_1-th and the r_2-th row.

Elementary operation matrices

Definition

$A^{(n,t,a)}$, $M^{(n,k,\alpha)}$ and $T^{(n,r_1,r_2)}$ are elementary operation matrices.

Lemma

$$A \sim B$$

if and only if

$$B = E_1 E_2 \cdots E_m A$$

for some elementary operation matrices $E_1, \ldots, E_m$.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

subtract 1st row from 2nd, exchange 2nd and 3rd row, add 2nd row to 3rd, multiply 3rd row by $-1/2$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = M^{(3,3,-1/2)} A^{(3,3,2)} T^{(3,2,3)} S^{(3,2,1)} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where

$$S^{(3,2,1)} = M^{(3,1,-1)} A^{(3,2,1)} M^{(3,1,-1)}$$

Invertibility of elementary row operations

- If we exchange rows r_1 and r_2, and then exchange them again, we obtain the original matrix.

$$T^{(n,r_1,r_2)}T^{(n,r_1,r_2)} = I^{(n)}$$

- If we multiply r-th row by α, and then by $1/\alpha$, we obtain the original matrix.

$$M^{(n,r,1/\alpha)}M^{(n,r,\alpha)} = I^{(n)}$$

- If we add the a-th row to the t-th, and then subtract the a-th row from the t-th, we obtain the original matrix. Equivalently,
  - multiply the a-th row by -1,
  - add a-th row to the t-th one, and
  - multiply the a-th row by -1.

$$\left[M^{(n,a,-1)}A^{(n,t,a)}M^{(n,a,-1)}\right]A^{(n,t,a)} = I^{(n)}$$

Invertibility of row equivalence

Corollary

$$A \sim B$$

if and only if

$$A = E_1 E_2 \cdots E_m B$$

for some elementary operation matrices $E_1, \ldots, E_m$.

Inverse to elementary row operations

Definition

For an elementary operation matrix E, let

$$E^{-1} = \begin{cases} M^{(n,a,-1)} A^{(n,t,a)} M^{(n,a,-1)} & \text{if } E = A^{(n,t,a)} \\ M^{(n,r,1/\alpha)} & \text{if } E = M^{(n,r,\alpha)} \\ T^{(n,r_1,r_2)} & \text{if } E = T^{(n,r_1,r_2)} \end{cases}$$

So that

$$E^{-1}E = I = EE^{-1}$$

Lemma

For an elementary operation matrix E,

$$Ex = b \text{ if and only if } x = E^{-1}b.$$

Matrix division?

For real numbers:

- $\frac{\beta}{\alpha}$ is the solution to $\alpha x = \beta$.
- Except if $\alpha = 0$: no or infinitely many solutions.

For matrices:

- Solution to $AX = B$ or $XA = B$?
- "Non-zero" elements: matrices A for that there always exists exactly one solution?

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix}$$

Equivalent to two systems of linear equations with the same left-hand sides:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} X_{1,1} \\ X_{2,1} \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} X_{1,2} \\ X_{2,2} \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$$

They can be solved at the same time:

$$\text{RREF} \left(\begin{array}{cc|cc} 1 & 2 & 8 & 5 \\ 3 & 4 & 20 & 13 \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & 1 \end{array} \right) \Rightarrow X = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

Consequences for matrix division

Let A be $n \times m$ matrix. The following are equivalent:

- $AX = B$ has unique solution for every $n \times p$ matrix B.
- $Ax = b$ has unique solution for every vector b

When is this the case?

- A must be square; otherwise there are either
  - "too many" equations: no solution, or
  - "too few" equations: infinitely many solutions.
- Not sufficient.

Equivalent characterizations of regularity

The following claims are equivalent for any $n \times n$ matrix A.

1. For every b, $Ax = b$ has exactly one solution.
2. $Ax = 0$ has only one solution ($x = 0$).
3. $\mathrm{RREF}(A) = I$
4. $\mathrm{rank}(A) = n$
5. $A \sim I$
6. A is a product of elementary operation matrices.
7. For every b, $Ax = b$ has a solution.

Proofs

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$$

$$1 \Rightarrow 7 \Rightarrow 3, \; 3 \Leftrightarrow 4$$

1. For every b, $Ax = b$ has exactly one solution.
2. $Ax = 0$ has only one solution ($x = 0$).

Trivial.

Proofs

$$❶ \Rightarrow ❷ \Rightarrow ❸ \Rightarrow ❺ \Rightarrow ❻ \Rightarrow ❶$$

$$❶ \Rightarrow ❼ \Rightarrow ❸, ❸ \Leftrightarrow ❹$$

2. $Ax = 0$ has only one solution ($x = 0$).
3. $\text{RREF}(A) = I$

If $\text{RREF}(A) \neq I$, then not all columns of A are basis, and

$$(A|0) \sim (\text{RREF}(A)|0)$$

has infinitely many solutions.

Proofs

$$❶ \Rightarrow ❷ \Rightarrow ❸ \Rightarrow ❺ \Rightarrow ❻ \Rightarrow ❶$$

$$❶ \Rightarrow ❼ \Rightarrow ❸, ❸ \Leftrightarrow ❹$$

3. $\mathrm{RREF}(A) = I$
5. $A \sim I$

Trivial, since $A \sim \mathrm{RREF}(A)$.

Proofs

$$❶ \Rightarrow ❷ \Rightarrow ❸ \Rightarrow ❺ \Rightarrow ❻ \Rightarrow ❶$$

$$❶ \Rightarrow ❼ \Rightarrow ❸,\ ❸ \Leftrightarrow ❹$$

5. $A \sim I$
6. A is a product of elementary operation matrices.

As we observed before, $A \sim I$ if and only if

$$A = E_1 E_2 \cdots E_m I = E_1 E_2 \cdots E_m$$

for some elementary operation matrices $E_1, \ldots, E_m$.

Proofs

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$$

$$1 \Rightarrow 7 \Rightarrow 3, \; 3 \Leftrightarrow 4$$

6. A is a product of elementary operation matrices.
1. For every b, $Ax = b$ has exactly one solution.

Let $A = E_1 \cdots E_m$ for elementary operation matrices $E_1, \ldots, E_m$.

$$\begin{aligned} E_1 E_2 E_3 \cdots E_m x &= b \text{ if and only if} \\ E_2 E_3 \cdots E_m x &= E_1^{-1} b \text{ if and only if} \\ E_3 \cdots E_m x &= E_2^{-1} E_1^{-1} b \text{ if and only if} \\ &\cdots \\ x &= E_m^{-1} \cdots E_3^{-1} E_2^{-1} E_1^{-1} b \end{aligned}$$

Proofs

$$❶ \Rightarrow ❷ \Rightarrow ❸ \Rightarrow ❺ \Rightarrow ❻ \Rightarrow ❶$$

$$❶ \Rightarrow ❼ \Rightarrow ❸,\ ❸ \Leftrightarrow ❹$$

1. For every b, $Ax = b$ has exactly one solution.
7. For every b, $Ax = b$ has a solution.

Trivial.

Proofs

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$$

$$1 \Rightarrow 7 \Rightarrow 3,\ 3 \Leftrightarrow 4$$

7. For every b, $Ax = b$ has a solution.
3. $\mathrm{RREF}(A) = I$

If $\mathrm{RREF}(A) \neq I$, the last row of $\mathrm{RREF}(A)$ is 0, and

$$\mathrm{RREF}(A)x = e_n$$

has no solution. We have $A = E_1 \cdots E_m \mathrm{RREF}(A)$ for some elementary operation matrices $E_1, \ldots, E_m$. Consequently,

$$(\mathrm{RREF}(A)|e_n) \sim (A|E_1 \cdots E_m e_n),$$

and thus $Ax = E_1 \cdots E_m e_n$ has no solution.

Proofs

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$$

$$1 \Rightarrow 7 \Rightarrow 3, \; 3 \Leftrightarrow 4$$

3. $\mathrm{RREF}(A) = I$
4. $\mathrm{rank}(A) = n$

Trivial from the definition of rank.

Regular matrices

Definition

A square matrix A is regular if it satisfies any of the described equivalent conditions.

Further remarks on regular matrices

Lemma

$A \sim B$ iff there exists a regular matrix Q such that $A = QB$.

Lemma

Let A and B be $n \times n$ matrices. Then AB is regular if and only if both A and B are regular.

Proof.

$\Leftarrow$ If A, B are products of elementary operation matrices, then so is AB.

$\Rightarrow$ If B is not regular, then there exists $x \neq 0$ such that $Bx = 0$, and thus $ABx = 0$; hence AB is not regular.

If B is regular and A is not regular, then there exists $y \neq 0$ such that $Ay = 0$, and x such that $Bx = y$ (clearly, $x \neq 0$). Hence, again, $ABx = 0$ implying that AB is not regular.

Matrix inverse

Definition

Let A be a square matrix.

- If $AC = I$, then C is a right inverse to A.
- If $DA = I$, then D is a left inverse to A.

Lemma

If A has both a left inverse D and a right inverse C, then $C = D$. Hence, if both left and right inverses exist, they are unique.

Proof.

$$D = DI = D(AC) = (DA)C = IC = C$$

$\square$

Inverse and regularity

Lemma

If a square matrix A has a left or right inverse, then A is regular.

Proof.

I is regular, so if $XY = I$, then X and Y are regular. □

Existence of an inverse

Lemma

If A is regular, then it has both a left and a right inverse.

Proof.

Since A is regular, $A = E_1 \cdots E_m$ for some elementary operation matrices $E_1, \ldots, E_m$. Then,

$$E_m^{-1} \cdots E_1^{-1} A = I = A E_m^{-1} \cdots E_1^{-1}.$$

□

Existence of an inverse, another way

Lemma

If A is regular, then it has both a left and a right inverse.

Proof.

If A is regular, then there exist column matrices $c_1, \ldots, c_n$ such that $Ac_1 = e_1$, $Ac_2 = e_2$, $\ldots$, $Ac_n = e_n$. Hence, $AC = I$, where $C = (c_1|c_2|\ldots|c_n)$.
Note that

$$A(I - CA) = A - (AC)A = A - A = O.$$

Since A is regular, $AX = O$ if and only if $X = O$. Hence, $I - CA = O$ and $CA = I$. $\square$

Matrix inverse: summary

The following claims are equivalent for a square matrix A:

1. A is regular
2. A has a left inverse
3. A has a right inverse
4. A has a unique left inverse and a unique right inverse, and they are equal.

Definition

For a regular square matrix A, the inverse A^{-1} is the matrix satisfying

$$AA^{-1} = A^{-1}A = I.$$

Inverse, regularity and matrix multiplication

For regular $n \times n$ matrices A and B:

- A^{-1} is regular, and A is its inverse.
- $(AB)^{-1} = B^{-1}A^{-1}$
  - Since $(AB)\left[B^{-1}A^{-1}\right] = A\left[BB^{-1}\right]A^{-1} = AIA^{-1} = AA^{-1} = I$.
- Let C and D be $n \times m$ matrices. Then

$$AC = AD \text{ if and only if } C = D.$$

  - $AC = AD$ implies $A^{-1}AC = A^{-1}AD$
- For $m \times n$ matrices C' and D',

$$C'A = D'A \text{ if and only if } C' = D'.$$

- $AX = C$ has unique solution $X = A^{-1}C$
- $XA = C'$ has unique solution $X = C'A^{-1}$

Computing an inverse matrix

Lemma

For a regular matrix A,

$$RREF(A|I) = (I|A^{-1}).$$

Proof.

Solution to n systems of linear equations $Ac_1 = e_1, \ldots,$ $Ac_n = e_n$. $\square$

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

Hence,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$