

Spectral radius, symmetric and positive matrices

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April 28, 2016

1 Spectral radius

Definition 1. The spectral radius of a square matrix A is

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

For an $n \times n$ matrix A , let $\|A\| = \max\{|A_{ij}| : 1 \leq i, j \leq n\}$.

Lemma 1. If $\rho(A) < 1$, then

$$\lim_{n \rightarrow \infty} \|A^n\| = 0.$$

If $\rho(A) > 1$, then

$$\lim_{n \rightarrow \infty} \|A^n\| = \infty.$$

Proof. Recall that $A = CJC^{-1}$ for a matrix J in Jordan normal form and regular C , and that $A^n = CJ^nC^{-1}$. If $\rho(A) = \rho(J) < 1$, then J^n converges to the 0 matrix, and thus A^n converges to the zero matrix as well. If $\rho(A) > 1$, then J^n has a diagonal entry $(J^n)_{ii} = \lambda^n$ for an eigenvalue λ such that $|\lambda| > 1$, and if v is the i -th column of C and v' the i -th row of C^{-1} , then $v'A^n v = v'CJ^nC^{-1}v = e_i^T J^n e_i = \lambda^n$. Therefore, $\lim_{n \rightarrow \infty} |v'A^n v| = \infty$, and thus $\lim_{n \rightarrow \infty} \|A^n\| = \infty$. \square

2 Matrices with real eigenvalues

Lemma 2 (Schur). If all eigenvalues of a real $n \times n$ matrix A are real, then $A = QUQ^T$ for some upper-triangular matrix U and an orthogonal matrix Q .

Proof. We prove the claim by induction on n . Let λ be an eigenvalue of A and let $B' = v_1, \dots, v_k$ be an orthonormal basis of the space $\text{Ker}(A - \lambda I)$ of the eigenvectors of A for λ . Let us extend B' to an orthonormal basis $B = v_1, \dots, v_n$ of \mathbf{R}^n , and let $C = (v_1|v_2|\dots|v_n)$. Note that C is orthogonal and $C^T A C = \begin{pmatrix} \lambda I_k & X \\ 0 & A' \end{pmatrix}$ for some matrices X and A' . The characteristic polynomial of A is $\det(A - Ix) = (\lambda - x)^k \det(A' - Ix)$, and thus all eigenvalues of A' are also eigenvalues of A , and thus they are real. By the induction hypothesis, $D_0^T A' D_0 = U'$ for an upper-triangular matrix A' and an orthogonal matrix D_0 . Let $D = \begin{pmatrix} I_k & 0 \\ 0 & D_0 \end{pmatrix}$ and note that D is also orthogonal. We have

$$D^T C^T A C D = \begin{pmatrix} \lambda I_k & X D_0 \\ 0 & U' \end{pmatrix},$$

which is upper-triangular, and thus we can set $Q = CD$. □

3 Symmetric matrices

Lemma 3. *If a real matrix A is symmetric, then all its eigenvalues are real.*

Proof. Suppose that λ is an eigenvalue of A and let v be a corresponding eigenvector (possibly complex). Then $\lambda \langle v, v \rangle = \lambda v^T \bar{v} = (Av)^T \bar{v} = (v^T A^T) \bar{v} = (v^T A) \bar{v} = v^T (A \bar{v}) = v^T (\overline{Av}) = \bar{\lambda} v^T \bar{v} = \bar{\lambda} \langle v, v \rangle$, and thus $\lambda = \bar{\lambda}$ and λ is real. □

Corollary 4. *If a real matrix A is symmetric, then $A = QDQ^T$ for a diagonal matrix D and an orthogonal matrix Q ; i.e., A is diagonalizable and there exists an orthonormal basis formed by eigenvectors of A .*

Proof. By Lemma 2, we have $A = QUQ^T$ for an upper-triangular matrix A and an orthogonal matrix Q . Since A is symmetric, we have $A = A^T = (QUQ^T)^T = QU^T Q^T$, and since Q is regular, it follows that $U^T = U$. Hence, U is symmetric, and thus U is diagonal. It follows that columns of Q are eigenvectors of A , and since Q is orthogonal, they form an orthonormal basis. □

Lemma 5. *If A is a symmetric real matrix A , then $\max\{x^T A x : \|x\| = 1\}$ is the largest eigenvalue of A .*

Proof. Let $A = QDQ^T$ for a diagonal matrix D and an orthogonal matrix Q . Note that A and D have the same eigenvalues and that $\|Qx\| = \|x\|$

for every x , and since Q is regular, it follows that $\max\{x^T Ax : \|x\| = 1\} = \max\{x^T Dx : \|x\| = 1\}$. Therefore, it suffices to show that $\max\{x^T Dx : \|x\| = 1\}$ is the largest eigenvalue of D . Let $d_1 \geq d_2 \geq \dots \geq d_n$ be the diagonal entries of D , which are also its eigenvalues. Then $x^T Dx = \sum_{i=1}^n d_i x_i^2 \leq d_1 \sum_{i=1}^n x_i^2 = d_1 \|x\|^2 = d_1$ for every x such that $\|x\| = 1$, and $e_1^T D e_1 = d_1$. The claim follows. \square

4 Positive matrices

A matrix A is non-negative if all its entries are non-negative, and it is positive if all its entries are positive.

Lemma 6. *If A is a positive matrix, $\rho(A) = 1$, and λ is an eigenvalue of A with $|\lambda| = 1$, then the real part of λ is positive.*

Proof. Suppose for a contradiction that the real part of λ is non-positive. Choose $\varepsilon > 0$ such that $A_{ii} > \varepsilon$ for every i . Then $|\lambda - \varepsilon| > 1$. Choose $0 < \delta < 1$ such that $\delta|\lambda - \varepsilon| > 1$.

Let $A_1 = \delta(A - \varepsilon I)$ and $A_2 = \delta A$. Note that $\delta(\lambda - \varepsilon)$ is an eigenvalue of A_1 , and thus $\rho(A_1) > 1$. On the other hand, $\rho(A_2) = \delta\rho(A) < 1$. By Lemma 1, $\lim_{n \rightarrow \infty} \|A_2^n\| = 0$ and $\lim_{n \rightarrow \infty} \|A_1^n\| = \infty$. However, each entry of A_1 is at most as large as each entry of A_2 , and A_1 is a positive matrix, and thus $(A_1^n)_{ij} \leq (A_2^n)_{ij}$ for all i, j, n . This is a contradiction. \square

Theorem 7 (Perron-Frobenius). *Let A be a non-negative square matrix. If some power of A is positive, then $\rho(A)$ is an eigenvalue of A and all other eigenvalues of A have absolute value strictly less than $\rho(A)$.*

Proof. The claim is trivial if $\rho(A) = 0$, hence assume that $\rho(A) > 0$. Let $m_0 \geq 1$ be an integer such that A^{m_0} is positive. Since A is non-negative, A^m is positive for every $m \geq m_0$. Suppose that λ is an eigenvalue of A with $|\lambda| = \rho(A)$. Let $A_1 = A/\rho(A)$ and note that $\rho(A_1) = 1$ and $\lambda_1 = \lambda/\rho(A)$ is an eigenvalue of A_1 with $|\lambda_1| = 1$.

If $\lambda_1 \neq 1$, then there exists $m \geq m_0$ such that the real part of λ_1^m is non-positive. But λ_1^m is an eigenvalue of the positive matrix A_1^m with $\rho(A_1^m) = |\lambda_1^m| = 1$, which contradicts Lemma 6. Therefore, $\lambda_1 = 1$, and thus $\lambda = \rho(A)$ and A has no other eigenvalues with absolute value $\rho(A)$. \square

Lemma 8. *Let A be a non-negative square matrix such that some power of A is positive, with $\rho(A) = 1$. If v is a non-negative vector and $(Av)_i \geq v_i$ for every i , then $Av = v$.*

Proof. Let $m \geq 1$ be an integer such that A^m is positive. Suppose for a contradiction that $Av \neq v$, and thus $Av - v$ is non-negative and non-zero. Since A^m is positive, we have that $A^m(Av - v)$ is positive. Thus, there exists $\delta > 1$ such that $A^m(Av - \delta v)$ is still positive, and thus $(A^{m+1}v)_i \geq \delta(A^m v)_i$ for all i . Since A is non-negative, it follows by multiplying the inequality by A that $(A^{m+2}v)_i \geq \delta(A^{m+1}v)_i$ for all i . Combining these inequalities, $(A^{m+2}v)_i \geq \delta^2(A^m v)_i$ for all i . Similarly, $(A^{m+n}v)_i \geq \delta^n(A^m v)_i$ for all $n \geq 0$ and all i . Consider any β such that $1 < \beta < \delta$, and let $B = A/\beta$. Then $(B^{m+n}v)_i \geq (\delta/\beta)^n(B^m v)_i$ for all $n \geq 0$ and all i , and thus $\lim_{n \rightarrow \infty} \|B^n\| = \infty$. However, $\rho(B) = 1/\beta < 1$, which contradicts Lemma 1. Therefore, $Av = v$. \square

Lemma 9. *Let A be a non-negative $n \times n$ matrix. If some power of A is positive, then the algebraic multiplicity of $\rho(A)$ is one and there exists a positive eigenvector for $\rho(A)$.*

Proof. If $\rho(A) = 0$, then by considering the Jordan normal form of A , we conclude that $A^n = 0$, which contradicts the assumption that some power of A is positive. Hence, $\rho(A) > 0$. Without loss of generality, we can assume that $\rho(A) = 1$, as otherwise we divide A by $\rho(A)$ first. Let v be an eigenvector for 1, and let w be the vector such that $w_i = |v_i|$ for all i . We have $(Aw)_i \geq |(Av)_i| = |v_i| = w_i$ for all i , and by Lemma 8, it follows that $Aw = w$.

Let $m \geq 1$ be an integer such that A^m is positive. We have $A^m w = w$, and since w is non-negative, the vector $A^m w = w$ is positive. Thus, w is actually positive.

Suppose now for contradiction that the algebraic multiplicity of $\rho(A)$ is greater than 1. By considering the Jordan normal form of A , it follows that there exists a non-zero vector z linearly independent on w such that either $Az = z$, or $Az = z + w$. In the former case, there exists a non-zero vector $z' = w + \alpha z$ for some α such that z' is non-negative, but at least one coordinate of z' is 0. However, $Az' = z'$, and thus $A^m z' = z'$, and $A^m z'$ is positive, which is a contradiction. In the latter case, choose $\alpha > 0$ so that $w' = z + \alpha w$ is positive. Then $(Aw')_i = (z + (\alpha + 1)w)_i > w'_i$ for all i , which contradicts Lemma 8. \square

A square matrix is stochastic if each of its columns has sum equal to 1. Let j denote the row vector of all ones. Note that A is stochastic if and only if $jA = j$, and thus j^T is an eigenvector of A^T for the eigenvalue 1.

Lemma 10. *Let A be a non-negative square stochastic matrix such that some power of A is positive. Then there exists a unique positive vector v*

with $Av = v$ such that $juv = 1$. Furthermore, for any vector w such that $juw = 1$, we have

$$\lim_{k \rightarrow \infty} A^k w = v.$$

Proof. By Theorem 7 and Lemma 9, $\rho(A)$ is an eigenvalue of A with algebraic multiplicity 1, and the absolute value of any other eigenvalue is strictly less than $\rho(A)$, and there exists a positive vector v such that $Av = \rho(A)v$. Choose v so that $juv = 1$. We have $\rho(A) = \rho(A)juv = j(Av) = (jA)v = juv = 1$, and thus $Av = v$ and $\rho(A) = 1$.

Let J be the matrix in Jordan normal form such that $A = CJC^{-1}$, such that $J_{11} = 1$ and all other diagonal entries of J are strictly smaller than 1, and $C_{*,1} = v$. Let z be the first row of C^{-1} . We have $zA = zCJC^{-1} = e_1^T JC^{-1} = e_1^T C^{-1} = z$, and thus $A^T z^T = z^T$. Therefore, z^T is an eigenvector of A^T for eigenvalue 1. Note that the eigenvalues of A^T are the same as the eigenvalues of A , with the same algebraic multiplicities. Hence, 1 has multiplicity 1 as an eigenvalue of A^T , and thus the corresponding eigenvector is unique up to scalar multiplication. It follows that z is a multiple of j . Since z is the first row of C^{-1} and v the first column of C , we have $zv = 1$, and since $juv = 1$, it follows that $z = j$.

We have $\lim_{k \rightarrow \infty} J^k = e_1 e_1^T$, and thus $\lim_{k \rightarrow \infty} A^k w = C e_1 e_1^T C^{-1} w = v z w = v j w = v$. \square

Example 1. Let G be a connected non-bipartite graph. Start in a vertex v_1 of G , walk to its neighbor chosen uniformly at random, walk again to a neighbor of the target vertex chosen uniformly at random, etc.

Let $V(G) = \{v_1, \dots, v_n\}$. Let $p_{i,k}$ denote the probability that after k steps, we are in the vertex v_i , and let $p_k = (p_{1,k}, p_{2,k}, \dots, p_{n,k})^T$. So, $p_0 = e_1$. Furthermore, $p_{k+1} = Ap_k$, where A is the matrix such that $A_{i,j} = \frac{1}{\deg(v_j)}$ if $v_i v_j \in E(G)$ and $A_{i,j} = 0$ otherwise. Therefore, $p_k = A^k e_1$.

Since G is connected and not bipartite, there exists k_0 such that A^{k_0} is positive. By Lemma 10, we have $\lim_{k \rightarrow \infty} A^k e_1 = p$ for the unique positive vector p such that $Ap = p$ and $jp = 1$. Observe that this is true for $p = \frac{1}{2|E(G)|} (\deg(v_1), \dots, \deg(v_n))^T$.

Therefore, after many steps, the probability that the walk ends in a vertex v_i approaches $\frac{\deg(v_i)}{2|E(G)|}$.

For a directed graph G of links between webpages, the corresponding eigenvector p gives PageRank, which is one of the factors used to measure the importance of a webpage.