Determinants

Zdeněk Dvořák

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Definition 1. *Let A be an $n \times n$ matrix. The* <u>*determinant*</u> *of A is*

$$\det(A) = \sum_{\pi\text{: permutation of } \{1,\ldots,n\}} sgn(\pi) A_{1,\pi(1)} A_{2,\pi(2)} \ldots A_{n,\pi(n)}.$$

- Determinant of an upper-triangular matrix is the product of its diagonal elements.
- Adding a linear combination of rows to another row does not change the determinant.
- Determinant is linear in each row.
- Swapping two rows changes the sign of the determinant.

1 Further properties of determinants

Lemma 1. *For an $n \times n$ matrix A, $\det(A) \neq 0$ if and only if A is regular.*

Proof. By inspection of the Gauss-Jordan elimination algorithm, $\det(A) = \alpha \det(\mathrm{RREF}(A))$ for some $\alpha \neq 0$. Hence, $\det(A) \neq 0$ if and only if all diagonal entries of $\mathrm{RREF}(A)$ are non-zero, which is equivalent to A being regular. □

Lemma 2. *For any $n \times n$ matrix, $\det(A) = \det(A^T)$.*

Proof.

$$\begin{aligned}
\det(A^T) &= \sum_{\pi} \operatorname{sgn}(\pi)(A^T)_{1,\pi(1)} \dots (A^T)_{n,\pi(n)} \\
&= \sum_{\pi} \operatorname{sgn}(\pi) A_{\pi(1),1} \dots A_{\pi(n),n} \\
&= \sum_{\pi} \operatorname{sgn}(\pi) A_{1,\pi^{-1}(1)} \dots A_{n,\pi^{-1}(n)} \\
&= \sum_{\pi} \operatorname{sgn}(\pi^{-1}) A_{1,\pi^{-1}(1)} \dots A_{n,\pi^{-1}(n)} \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) A_{1,\sigma(1)} \dots A_{n,\sigma(n)} \\
&= \det(A).
\end{aligned}$$

□

Hence, everything we proved for the effect of row operations on the determinant also holds for column operations.

Definition 2. *For an $n \times m$ matrix A and integers i and j, let A^{ij} denote the $(n-1) \times (m-1)$ matrix obtained from A by removing the i-th row and the j-th column.*

Lemma 3 (Recursive formula for determinant)**.** *Let A be an $n \times n$-matrix, and let $i \in \{1, \dots, n\}$. Then*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(A^{ij}).$$

Proof. Suppose first that the i-th row contains only one non-zero entry A_{ij}. Let B be the matrix obtained from A by swapping the i-th row with the $(i-1)$-st, then with $(i-2)$-nd, …, and the j-th column with the $(j-1)$-st, …, so that $B_{1,\star} = (A_{ij}, 0, 0, \dots)$ and $B^{11} = A^{ij}$. Note that $\det(B) = (-1)^{i+j} \det(A)$. Furthermore, in the definition of $\det(B)$, only the terms with $\pi(1) = 1$ contribute a non-zero amount to the determinant, and thus $\det(B) = B_{11} \det(B^{11})$. It follows that $\det(A) = (-1)^{i+j} A_{ij} \det(A^{ij})$.

In general, the formula then follows from the linearity of the determinant in the i-th row. □

Example 1. *Determine*

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

$$\begin{aligned}
\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} &= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{pmatrix} - \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \\
&= \left[-\det\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} + \det\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\right] + \det\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \\
&= [-1+3] + 2 = 4
\end{aligned}$$

Lemma 4. *For any $n \times n$ matrix A and elementary operation matrix Q, $\det(QA) = \det(Q)\det(A)$.*

Proof. Let us distinguish the cases:

- If Q is the matrix of the operation of the addition of the r-th row to the s-th row, then $Q_{ii} = 1$ for $i = 1, \ldots, n$, $Q_{s,r} = 1$, and all other entries of Q are 0. Hence, Q is either upper triangular or lower triangular, and thus $\det(Q)$ is the product of its diagonal entries, which is 1.

  On the other hand, QA is obtained from A by adding the r-th row to the s-th row, and thus $\det(QA) = \det(A) = \det(Q)\det(A)$.

- If Q is the matrix of the operation of multiplication of the r-th row by a non-zero constant α, then $Q_{ii} = 1$ for $i \neq r$, $Q_{rr} = \alpha$ and all other entries of Q are 0, and thus $\det(Q) = \alpha$.

  On the other hand, QA is obtained from A by multiplying the r-th row by α, and thus $\det(QA) = \alpha\det(A) = \det(Q)\det(A)$.

- If Q is the matrix of the operation of exchanging the r-th row and the s-th row, then Q is obtained from I by exchanging the r-th row and the s-th row, and thus $\det(Q) = -\det(I) = -1$.

  On the other hand, QA is obtained from A by exchanging the r-th row and the s-th row, and thus $\det(QA) = -\det(A) = \det(Q)\det(A)$.

□

Lemma 5. *For any $n \times n$ matrices A and B,*

$$\det(AB) = \det(A)\det(B).$$

Proof. If A is not regular, then $\det(A) = 0$, and AB is not regular, and $\det(AB) = 0 = \det(A)\det(B)$.

If A is regular, then $A = Q_1 \dots Q_n$ for some elementary operation matrices $Q_1, \dots, Q_n$. Hence,

$$\begin{aligned}\det(AB) &= \det(Q_1 Q_2 \dots Q_n B)\\ &= \det(Q_1)\det(Q_2 \dots Q_n B)\\ &= \det(Q_1)\dots\det(Q_n)\det(B)\\ &= \det(Q_1 \dots Q_n)\det(B)\\ &= \det(A)\det(B).\end{aligned}$$

□

Corollary 6. *For any regular matrix A, $\det(A^{-1}) = 1/\det(A)$.*

Proof. $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$. □

Corollary 7. *Determinant of an orthogonal matrix is either 1 or -1.*

Proof. If Q is orthogonal, then $QQ^T = I$, and thus $1 = \det(I) = \det(Q)\det(Q^T) = \det(Q)^2$. □

2 Determinants and systems of equations

Theorem 8 (Cramer's rule)**.** *Let A be a regular matrix. If $x = (x_1, \dots, x_n)^T$ satisfies $Ax = b$, and $A_{i\to b}$ is the matrix obtained from A by replacing the i-th column by b, then*

$$x_i = \frac{\det(A_{i\to b})}{\det(A)}.$$

Proof. Since $b = Ax = x_1 A_{\star,1} + \dots + x_n A_{\star,n}$, the linearity of the determinant in the i-th column implies that

$$\det(A_{i\to b}) = x_1 \det(A_{i\to A_{\star,1}}) + \dots + x_n \det(A_{i\to A_{\star,n}}).$$

However, if $i \neq j$, then $A_{i\to A_{\star,j}}$ has two identical columns, and thus $\det(A_{i\to A_{\star,j}}) = 0$. Therefore,

$$\det(A_{i\to b}) = x_i \det(A_{i\to A_{\star,i}}) = x_i \det(A).$$

□

Example 2. *Solve the system of equations*

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ 2x_1 + x_2 - x_3 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 8 \end{aligned}$$

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} = -1$$

$$x_1 = \frac{\det \begin{pmatrix} 5 & 1 & 1 \\ 0 & 1 & -1 \\ 8 & 2 & 1 \end{pmatrix}}{-1} = \frac{-1}{-1} = 1$$

$$x_2 = \frac{\det \begin{pmatrix} 1 & 5 & 1 \\ 2 & 0 & -1 \\ 3 & 8 & 1 \end{pmatrix}}{-1} = \frac{-1}{-1} = 1$$

$$x_3 = \frac{\det \begin{pmatrix} 1 & 1 & 5 \\ 2 & 1 & 0 \\ 3 & 2 & 8 \end{pmatrix}}{-1} = \frac{-3}{-1} = 3$$

Theorem 9. *If A is a regular matrix, then*

$$(A^{-1})_{ij} = (-1)^{i+j} \frac{\det(A^{ji})}{\det(A)}.$$

Proof. We have $AA^{-1} = I$, and thus $A(A^{-1})_{\star,j} = I_{\star,j} = e_j$. By Theorem 8,

$$(A^{-1})_{ij} = \frac{\det(A_{i \to e_j})}{\det(A)},$$

and $\det(A_{i \to e_j}) = (-1)^{i+j} \det(A^{ji})$ by Lemma 3. □

Example 3. *Determine the inverse to* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}$.

$$\det(A) = 2$$

$$
\begin{array}{lll}
\det(A^{11}) = 2 & \det(A^{21}) = 1 & \det(A^{31}) = 1 \\
\det(A^{12}) = -2 & \det(A^{22}) = 0 & \det(A^{32}) = 2 \\
\det(A^{13}) = -2 & \det(A^{23}) = -1 & \det(A^{33}) = 1
\end{array}
$$

Hence,

$$
A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -1 & 1 \\ 2 & 0 & -2 \\ -2 & 1 & 1 \end{pmatrix}.
$$

Corollary 10. *Let A be a regular $n \times n$ matrix with integer coefficients. Then A^{-1} has integral coefficients if and only if $|\det(A)| = 1$. Equivalently, the system $Ax = b$ has integral solution for all integral right-hand sides b if and only if $|\det(A)| = 1$.*

Proof. If $|\det(A)| = 1$, then the formula from Theorem 9 gives integral coefficients for A^{-1}. Conversely, if A^{-1} has integral coefficients, then $\det(A^{-1})$ is an integer, and since $\det(A)$ is also an integer and $\det(A)\det(A^{-1}) = 1$, it follows that $|\det(A)| = 1$.

If A^{-1} is integral, then $x = A^{-1}b$ is integral. Furthermore, in the system $Ax = e_j$, we have $x_i = (A^{-1}e_j)_i = (A^{-1})_{ij}$, and thus if the system $Ax = b$ has integral solution for $b = e_1, \ldots, e_n$, then A^{-1} is integral. □