Bilinear and quadratic forms

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1 Bilinear forms

Definition 1. *Let* $\mathbf{V}$ *be a vector space over a field* $\mathbf{F}$*. A function* $b : \mathbf{V} \times \mathbf{V} \to \mathbf{F}$ *is a* *bilinear form* *if*

$$b(u+v,w) = b(u,w) + b(v,w) \qquad b(u,v+w) = b(u,v) + b(u,w)$$
$$b(\alpha u,v) = \alpha b(u,v) \qquad b(u,\alpha v) = \alpha b(u,v)$$

for all $u,v,w \in \mathbf{V}$ *and* $\alpha \in \mathbf{F}$.

The bilinear form b *is* *symmetric* *if* $b(u,v) = b(v,u)$ *for all* $u,v \in \mathbf{V}$.

Remark: $b(o,v) = b(v,o) = 0$.
Examples:

- $b(x,y) = \langle x,y\rangle$ in $\mathbf{R}^n$ is bilinear and symmetric for any scalar product.
- $b((x_1,y_1),(x_2,y_2)) = x_1x_2 + 2x_1y_2 + 3y_1x_2 + 4y_1y_2$ is bilinear, but not symmetric.

Definition 2. *Let* $C = v_1,\ldots,v_n$ *be a basis of* $\mathbf{V}$ *and let* b *be a bilinear form on* $\mathbf{V}$*. The* *matrix of* b *with respect to* C *is*

$$[b]_C = \begin{pmatrix} b(v_1,v_1) & b(v_1,v_2) & \ldots & b(v_1,v_n) \\ b(v_2,v_1) & b(v_2,v_2) & \ldots & b(v_2,v_n) \\ & & \ldots & \\ b(v_n,v_1) & b(v_n,v_2) & \ldots & b(v_n,v_n) \end{pmatrix}.$$

Lemma 1. *Let* $C = v_1,\ldots,v_n$ *be a basis of* $\mathbf{V}$ *and let* b *be a bilinear form on* $\mathbf{V}$*. For any* $x,y \in \mathbf{V}$*, we have*

$$b(x,y) = [x]_C[b]_C[y]_C^T.$$

Proof. Let $[x]_C = (\alpha_1, \ldots, \alpha_n)$ and $[y]_C = (\beta_1, \ldots, \beta_n)$. We have

$$\begin{aligned} b(x,y) &= b(\alpha_1 v_1 + \ldots + \alpha_n v_n, \beta_1 v_1 + \ldots + \beta_n v_n) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j b(v_i, v_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \left([b]_C\right)_{i,j} \\ &= [x]_C [b]_C [y]_C^T. \end{aligned}$$

□

Remark: $[b]_C$ is the only matrix with this property. A bilinear form b is symmetric if and only if $[b]_C$ is a symmetric matrix.

Corollary 2. *Let* $\mathbf{V}$ *be a vector space over a field* $\mathbf{F}$. *Let* $C = v_1, \ldots, v_n$ *be a basis of* $\mathbf{V}$. *For every* $n \times n$ *matrix* M *over* $\mathbf{F}$, *there exists a unique bilinear form* $b : \mathbf{V} \times \mathbf{V} \to \mathbf{F}$ *such that* $b(v_i, v_j) = M_{i,j}$ *for* $1 \leq i, j \leq n$.

Proof. Define $b(x,y) = [x]_C M [y]_C^T$ and observe that b is bilinear. No other bilinear form with this property exists, since any bilinear form satisfying the assumptions has matrix M, which by Lemma 1 uniquely determines the values of the bilinear form. □

Example 1. *The bilinear form* $b((x_1, y_1), (x_2, y_2)) = x_1 x_2 + 2x_1 y_2 + 3y_1 x_2 + 4y_1 y_2$ *has matrix* $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *with respect to the standard basis;*

$$b((x_1, y_1), (x_2, y_2)) = (x_1, y_1) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Lemma 3. *Let* $B = v_1, \ldots, v_n$ *and* C *be two bases of* $\mathbf{V}$ *and let* b *be a bilinear form on* $\mathbf{V}$. *Let* $S = [id]_{B,C}$. *Then*

$$[b]_B = S^T [b]_C S.$$

Proof. We have

$$\begin{aligned} \left(S^T [b]_C S\right)_{i,j} &= e_i^T S^T [b]_C S e_j \\ &= [v_i]_B S^T [b]_C S [v_j]_B^T \\ &= [v_i]_C [b]_C [v_j]_C^T \\ &= b(v_i, v_j) = \left([b]_B\right)_{i,j}. \end{aligned}$$

□

2 Quadratic forms

Definition 3. *A function $f : \mathbf{V} \to \mathbf{F}$ is a quadratic form if there exists a bilinear form $b : \mathbf{V} \times \mathbf{V} \to \mathbf{F}$ such that $f(x) = b(x, x)$ for every $x \in \mathbf{V}$.*

Example 2.

$f((x, y)) = x^2 + 5xy + 4y^2$ is a quadratic form, since $f((x, y)) = b((x, y), (x, y))$ for the bilinear form $b((x_1, y_1), (x_2, y_2)) = x_1x_2 + 2x_1y_2 + 3y_1x_2 + 4y_1y_2$.

Also, $f((x, y)) = b_1((x, y), (x, y))$ for the symmetric bilinear form $b((x_1, y_1), (x_2, y_2)) = x_1x_2 + \frac{5}{2}(x_1y_2 + y_1x_2) + 4y_1y_2$.

Lemma 4. *Let $\mathbf{V}$ be a vector space over a field $\mathbf{F}$ whose characteristic is not 2. For every quadratic form f, there exists a unique symmetric bilinear form b such that $f(x) = b(x, x)$ for every $x \in \mathbf{V}$.*

Proof. Since f is quadratic, there exists a bilinear form b_0 such that $f(x) = b_0(x, x)$ for every $x \in \mathbf{V}$. Let $b(x, y) = \frac{1}{2}(b_0(x, y) + b_0(y, x))$. Then b is a symmetric bilinear form and $b(x, x) = b_0(x, x)$ for every $x \in \mathbf{V}$. Hence, b is a symmetric bilinear form such that $f(x) = b(x, x)$ for every $x \in \mathbf{V}$.

To show that b is unique, it suffices to note that

$$b(x, y) = \frac{b(x + y, x + y) - b(x, x) - b(y, y)}{2} = \frac{f(x + y) - f(x) - f(y)}{2}$$

whenever b is a symmetric bilinear form b satisfying $f(x) = b(x, x)$ for every $x \in \mathbf{V}$. □

Hence, if f is a quadratic form on $\mathbf{V}$ and C is a basis of $\mathbf{V}$, then

$$f(x) = [x]_C A [x]_C^T$$

for a unique symmetric matrix A. We write $[f]_C = A$. Also, by Lemma 3, if B is another basis of $\mathbf{V}$ and $S = [\mathrm{id}]_{B,C}$, then

$$[f]_B = S^T [f]_C S.$$

Lemma 5. *Let $f : \mathbf{V} \to \mathbf{F}$ be a quadratic form. Let $B_1 = v_1, \ldots, v_n$ and $B_2 = w_1, \ldots, w_n$ be two bases of $\mathbf{V}$ such that both $[f]_{B_1}$ and $[f]_{B_2}$ are diagonal matrices. Then $[f]_{B_1}$ and $[f]_{B_2}$ have the same number of positive entries.*

Proof. For $i \in \{1, 2\}$, let a_i be the number of positive entries of $[f]_{B_i}$, and suppose for a contradiction that $a_1 > a_2$. Let $I = \{i : f(v_i) > 0\}$, $J = \{i : f(w_i) \leq 0\}$. Let $\mathbf{U}_1 = \mathrm{span}(\{v_i : i \in I\})$ and $\mathbf{U}_2 = \mathrm{span}(\{w_i : i \in J\})$. Note that $\dim(\mathbf{U}_1) + \dim(\mathbf{U}_2) = a_1 + (\dim(\mathbf{V}) - a_2) > \dim(\mathbf{V})$, and thus there exists a non-zero vector $v \in \mathbf{U}_1 \cap \mathbf{U}_2$. However,

$$f(v) = [v]_{B_1}[f]_{B_1}[v]_{B_1}^T = \sum_{i \in I} ([v]_{B_1})_i^2 f(v_i) > 0$$
$$f(v) = [v]_{B_2}[f]_{B_2}[v]_{B_2}^T = \sum_{i \in J} ([v]_{B_2})_i^2 f(w_i) \leq 0$$

This is a contradiction. □

For integers a, b, and c, let $D(a, b, c)$ be the diagonal matrix with

$$D_{i,i} = \begin{cases} +1 & \text{for } i = 1, \ldots, a, \\ -1 & \text{for } i = a+1, \ldots, a+b, \\ 0 & \text{for } i = a+b+1, \ldots, a+b+c. \end{cases} .$$

Theorem 6 (Sylvester's law of inertia)**.** *If f is a quadratic form on a vector space $\mathbf{V}$ over real numbers of finite dimension, then there exist unique integers a, b, and c and a basis B of $\mathbf{V}$ such that $[f]_B = D(a, b, c)$.*

Proof. Let A be a matrix of f with respect to any basis C of $\mathbf{V}$. We need to find a regular matrix S (which will serve as the transition matrix from basis B to C) such that $S^T AS = D(a, b, c)$ for some a, b, and c. Perform Gaussian elimination, applying the same operations to the rows and columns of A, until we obtain a diagonal matrix with only $+1$, -1, or 0 on the diagonal.

For the uniqueness, suppose that $[f]_{B_1} = D(a_1, b_1, c_1)$ and $[f]_{B_2} = D(a_2, b_2, c_2)$ for some bases B_1 and B_2. Let $S = [\mathrm{id}]_{B_1,B_2}$; hence, $D(a_1, b_1, c_1) = S^T D(a_2, b_2, c_2)S$. Since S is regular, we have $\mathrm{rank}(D(a_1, b_1, c_1)) = \mathrm{rank}(D(a_2, b_2, c_2))$, and thus $c_1 = c_2$. Also, by Lemma 5, we have $a_1 = a_2$, and thus $b_1 = b_2$. □

Definition 4. *We say that a quadratic form f on a vector space $\mathbf{V}$ of finite dimension has signature (a, b, c) if there exists a basis B such that $[f]_B = D(a, b, c)$.*

Example 3. *Let* $A = \begin{pmatrix} -1 & 1 & -3 & -1 \\ 1 & 7 & -1 & 5 \\ -3 & -1 & -1 & 1 \\ -1 & 5 & 1 & 7 \end{pmatrix}$. *By performing simultaneous* *row and column operations, we have*

$$
\begin{aligned}
A &\to \begin{pmatrix} -1 & 1 & -3 & -1 \\ 0 & 8 & -4 & 4 \\ -3 & -1 & -1 & 1 \\ -1 & 5 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -3 & -1 \\ 0 & 8 & -4 & 4 \\ -3 & -4 & -1 & 1 \\ -1 & 4 & 1 & 7 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & -3 & -1 \\ 0 & 8 & -4 & 4 \\ 0 & -4 & 8 & 4 \\ -1 & 4 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 8 & -4 & 4 \\ 0 & -4 & 8 & 4 \\ -1 & 4 & 4 & 7 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 8 & -4 & 4 \\ 0 & -4 & 8 & 4 \\ 0 & 4 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 8 & -4 & 4 \\ 0 & -4 & 8 & 4 \\ 0 & 4 & 4 & 8 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 2 \\ 0 & -4 & 8 & 4 \\ 0 & 4 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & -2 & 8 & 4 \\ 0 & 2 & 4 & 8 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 6 & 6 \\ 0 & 2 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 6 & 6 \\ 0 & 2 & 6 & 8 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 6 & 6 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Hence, the signature of A is $(2,1,1)$.

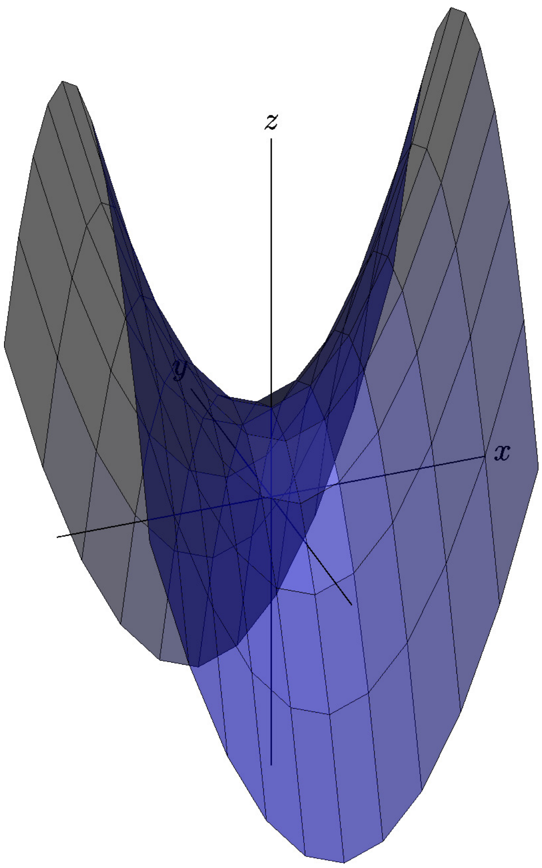
Corollary 7. *Let A be a real symmetric $n \times n$ matrix and let $f(x) = x^T Ax$ for $x \in \mathbf{R}^n$. If f has signature (a, b, c), then the sum of algebraic multiplicities of the positive eigenvalues of A is a, and the sum of algebraic multiplicities of the negative eigenvalues of A is b.*

Proof. Recall that since A is real and symmetric, we have $A = Q^{-1}DQ = Q^T DQ$ for a diagonal matrix D and an orthogonal matrix Q, where A and D have the same eigenvalues, equal to the diagonal entries of D. Equivalently, we have $[f]_{B_1} = D$, where B_1 is the basis formed by the columns of Q^T. Since the signature of A is (a, b, c), there exists a basis B_2 such that $[f]_{B_2} = D(a, b, c)$. The claim follows by Lemma 5. $\square$

3 Quadrics and conics

Definition 5. *For any $n \times n$ symmetric real matrix A, a real row vector b and a real number γ, the set $\{x \in \mathbf{R}^n : x^T Ax + bx + \gamma = 0\}$ is called a <u>quadric</u>. If $n = 2$, it is called a <u>conic</u>.*

Example 4. *The set $\{(x, y, z) \in \mathbf{R}^3 : x^2 - y^2 - z + 1 = 0\}$:*



Consider a quadric $C = \{x \in \mathbf{R}^n : x^T Ax + bx + \gamma = 0\}$. We have $A = Q^T DQ$ for an orthogonal matrix Q and a diagonal matrix D. Let $b' = bQ^T$ and $C' = \{y \in \mathbf{R}^n : y^T Dy + b'y + \gamma = 0\}$. Observe that $x \in C$ if and only if $Qx \in C'$, and thus thus the sets C' and C only differ by the isometry described by Q. Let (p, n, z) be the signature of A; without loss of generality, the first p entries of D are positive, the next n entries are negative and the last z are zeros. Furthermore, since C' is also equal to $\{y \in \mathbf{R}^n : y^T(-D)y - b'y - \gamma = 0\}$, we can assume that $p \geq n$.

For any vector d, let $b_d = 2d^T D + b'$, $\gamma_d = d^T Dd + b'd + \gamma$ and $C_d = \{v \in \mathbf{R}^n : v^T Dv + b_d v + \gamma_d\}$. Note that $v \in C_d$ if and only if $v + d \in C'$, and thus C_d is obtained from C' by shifting it by the vector d (another isometry). Choose the first $p + n$ coordinates of d so that the first $p + n$ coordinates of the vector $2d^T D$ are equal to the first $p+n$ coordinates of $-b'$ (the remaining coordinates of $2d^T D$ are always 0). Furthermore, if $z > 0$ and at least one of the last z entries of b' is not 0, we can choose the last z coordinates of d so that γ_d is 0.

Thus, we get the following.

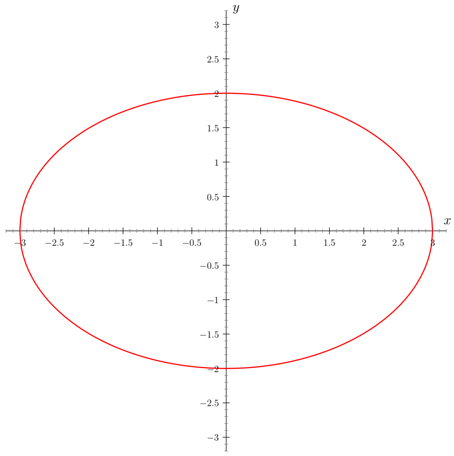
Lemma 8. *Every quadric is up to isometry equal to a quadric $\{x \in \mathbf{R}^n : x^T Ax + bx + \gamma = 0\}$ satisfying the following conditions:*

- *A is diagonal with the first p entries positive, the next n entries negative and the last z equal to 0 for some $p \geq n$,*
- *the first $p + n$ entries of b are equal to 0, and*
- *either $b = o$ or $\gamma = 0$.*

Example 5. *Classification of conics:*

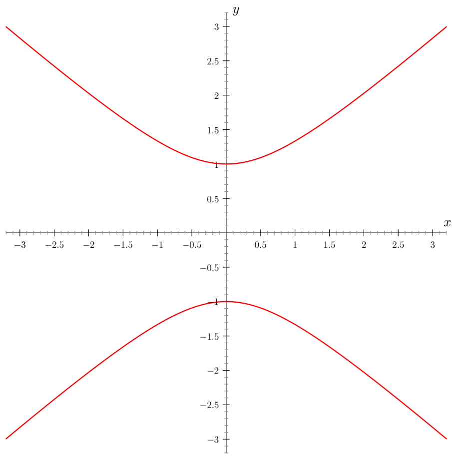
p = 2: $\alpha_1 x_1^2 + \alpha_2 x_2^2 = \gamma$ *for* $\alpha_1, \alpha_2 > 0$.

- *empty if* $\gamma < 0$,
- *the point* $(0, 0)$ *if* $\gamma = 0$,
- *an* <u>ellipse</u> *with axes* $\sqrt{\gamma/\alpha_1}$ *and* $\sqrt{\gamma/\alpha_2}$ *if* $\gamma > 0$.



$\mathbf{p = n = 1}$: *$\alpha_1 x_1^2 - \alpha_2 x_2^2 = \gamma$ for $\alpha_1, \alpha_2 > 0$.*

- *Two intersecting lines $|x_1| = \sqrt{\alpha_2/\alpha_1}|x_2|$ if $\gamma = 0$.*
- *<u>Hyperbola</u> if $\gamma \neq 0$.*

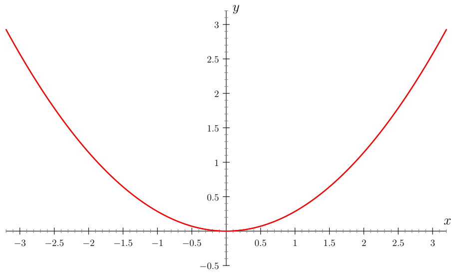


$\mathbf{p = z = 1,\ b = o}$: *$\alpha x_1^2 = \gamma$ for $\alpha > 0$.*

- *Empty, a line, or two parallel lines depending on γ.*

$\mathbf{p = z = 1,\ b \neq o,\ \gamma = 0}$: *$\alpha x_1^2 = \beta x_2$ for $\alpha, \beta \neq 0$.*

- *A <u>parabola</u>.*



$\mathbf{z = 2, \ b = o}$: $\gamma = 0$.

- *Empty or* $\mathbf{R}^2$ *depending on* γ.

$\mathbf{z = 2, \ b \neq o, \ \gamma = 0}$: $\beta_1 x_1 + \beta_2 x_2 = 0$.

- *A line.*

Similarly, we can classify quadrics in higher dimensions, based on their signatures.