Reminders: subspaces

Let $\mathbf{V}$ be a vector space over field $\mathbf{F}$.

Definition

A subset $\mathbf{U}$ of $\mathbf{V}$ is a subspace if it together with the operations of $\mathbf{V}$ forms a vector space.

Lemma

$\mathbf{U} \subseteq \mathbf{V}$ is a subspace if and only if

- *$o \in \mathbf{U}$, and*
- *for all $u, v \in \mathbf{U}$ and $\alpha \in \mathbf{F}$,*
    - *$u + v \in \mathbf{U}$, and*
    - *$\alpha v \in \mathbf{U}$.*

Reminders: linear functions

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.

Definition

A function $f : \mathbf{U} \to \mathbf{V}$ is linear if

- for every $u_1, u_2 \in \mathbf{U}$,

$$f(u_1 + u_2) = f(u_1) + f(u_2), \text{ and}$$

- for every $u \in \mathbf{U}$ and $\alpha \in \mathbf{F}$,

$$f(\alpha u) = \alpha f(u).$$

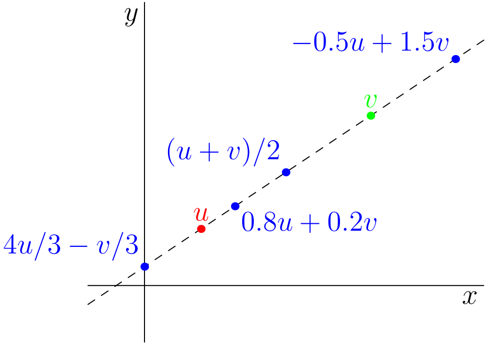
Affine sets: motivation

- $\{(x, y, z) : 3x - 3y + z = 0\}$ is a subspace
- $\{(x, y, z) : 3x - 3y + z = 2\}$ is not a subspace

Observation

The set of solutions to system $Ax = b$ is a subspace if and only if $b = o$.

Affine combinations



Definition

A linear combination $\alpha_1 v_1 + \ldots + \alpha_k v_k$ is affine if $\alpha_1 + \ldots + \alpha_n = 1$.

Affine sets

Let $\mathbf{V}$ be a vector space over field $\mathbf{F}$.

Definition

A set $U \subseteq \mathbf{V}$ is affine if every affine combination of elements of U belongs to U.

- Any subspace is an affine set.
- A line in Euclidean plane is an affine set.

Affine sets as shifted subspaces

Lemma

Let $U \subseteq \mathbf{V}$, $U \neq \emptyset$. The following claims are equivalent.

1. *U is affine.*
2. $\alpha x + (1-\alpha)y, x + y - z \in U$ *for all* $x, y, z \in U$ *and* $\alpha \in \mathbf{F}$.
3. *The set* $U - a = \{u - a : u \in U\}$ *is a subspace for all* $a \in U$.
4. *There exists a subspace* $\mathbf{W}$ *and* $b \in \mathbf{V}$ *such that* $U = \mathbf{W} + b = \{w + b : w \in \mathbf{W}\}$.

Proof.

1 $\Rightarrow$ 2 $\alpha x + (1-\alpha)y$ and $x + y - z$ are affine combinations.

Affine sets as shifted subspaces

Lemma

Let $U \subseteq \mathbf{V}$, $U \neq \emptyset$. The following claims are equivalent.

2. $\alpha x + (1-\alpha)y, x + y - z \in U$ *for all* $x, y, z \in U$ *and* $\alpha \in \mathbf{F}$.
3. *The set* $U - a = \{u - a : u \in U\}$ *is a subspace for all* $a \in U$.

Proof.

2 $\Rightarrow$ 3 Let $r, s \in U - a$, $\alpha \in \mathbf{F}$.

- Since $a \in U$, $o = a - a \in U - a$.
- Since $r, s \in U - a$, we have $r + a, s + a \in U$, and

$$\begin{aligned} r + s + a &= (r + a) + (s + a) - a \in U \\ \alpha r + a &= \alpha(r + a) + (1 - \alpha)a \in U, \end{aligned}$$

and thus $r + s, \alpha r \in U - a$.

□

Affine sets as shifted subspaces

Lemma

Let $U \subseteq \mathbf{V}$, $U \neq \emptyset$. The following claims are equivalent.

1. *U is affine.*
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4. *There exists a subspace* $\mathbf{W}$ *and* $b \in \mathbf{V}$ *such that* $U = \mathbf{W} + b = \{w + b : w \in \mathbf{W}\}$.

Proof.

3 $\Rightarrow$ 4 Choose $b \in U$ arbitrarily and let $\mathbf{W} = U - b$.

Affine sets as shifted subspaces

Lemma

Let $U \subseteq \mathbf{V}$, $U \neq \emptyset$. The following claims are equivalent.

1. *U is affine.*
2. $\alpha x + (1-\alpha)y, x + y - z \in U$ *for all* $x, y, z \in U$ *and* $\alpha \in \mathbf{F}$.
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4. *There exists a subspace* $\mathbf{W}$ *and* $b \in \mathbf{V}$ *such that* $U = \mathbf{W} + b = \{w + b : w \in \mathbf{W}\}$.

Proof.

4 $\Rightarrow$ 1 Suppose that $u_1, \ldots, u_k \in U$ and $\alpha_1 + \ldots + \alpha_k = 1$. Then $u_1 - b, \ldots, u_k - b \in \mathbf{W}$, and by linearity,

$$\alpha_1 u_1 + \ldots + \alpha_k u_k - b = \alpha_1(u_1 - b) + \ldots + \alpha_k(u_k - b) \in \mathbf{W}.$$

Hence, $\alpha_1 u_1 + \ldots + \alpha_k u_k \in \mathbf{W} + b = U$.

Computations and concepts in affine sets

Since affine sets are just shifted subspaces ($U = \mathbf{W} + b$), we can:

- Define the dimension of affine set $\dim(U) = \dim(\mathbf{W})$.
- Describe U by giving b and a basis of $\mathbf{W}$.
- Describe elements of U by coordinates in $\mathbf{W}$.

Reminder: characteristic 2

Definition

A field $\mathbf{F}$ has characteristic 2 if $1 + 1 = 0$.

Simpler affinity test

Let $\mathbf{V}$ be a vector space over field $\mathbf{F}$

Lemma

Suppose that $\mathbf{F}$ *<u>does not have characteristic 2</u>. A non-empty set* $U \subseteq \mathbf{V}$ *is affine if and only if for all* $x, y \in U$ *and* $\alpha \in \mathbf{F}$, $\alpha x + (1 - \alpha) y \in U$.

Proof.

$\Rightarrow$ Trivial.

$\square$

Simpler affinity test

Let $\mathbf{V}$ be a vector space over field $\mathbf{F}$

Lemma

Suppose that $\mathbf{F}$ *does not have characteristic 2. A non-empty set* $U \subseteq \mathbf{V}$ *is affine if and only if for all* $x, y \in U$ *and* $\alpha \in \mathbf{F}$*,* $\alpha x + (1 - \alpha) y \in U$*.*

Proof.

$\Leftarrow$ It suffices to prove $x + y - z \in U$ for all $x, y, z \in U$.
Let $w = (1+1)^{-1} x + (1+1)^{-1} y$.

- Since $(1+1)^{-1} + (1+1)^{-1} = (1+1) \cdot (1+1)^{-1} = 1$, we have $w \in U$.
- Since $(1+1) + (-1) = 1$, we have $(1+1)w - z \in U$.
- $(1+1)w - z = (1+1)(1+1)^{-1}(x+y) - z = x + y - z$.

$\square$

Affinity of solution sets

Let A be an $n \times m$ matrix with coefficients from field $\mathbf{F}$.

Lemma

The set of solutions to system $Ax = b$ is affine.

Proof.

This is trivial if there is no solution. Let x_0 be a solution.

- Recall that $\mathrm{Ker}(A)$ is the set of solutions of $Ax = 0$.
- If $Ax = b$, then $A(x - x_0) = Ax - Ax_0 = b - b = 0$, hence $x - x_0 \in \mathrm{Ker}(A)$.
- The set of solutions is $\mathrm{Ker}(A) + x_0$.

$\square$

Changing the right-hand side only "shifts" the set of solutions.

Subspaces and kernels

Let $\mathbf{V}$ be a vector space over field $\mathbf{F}$.

Lemma

A set $U \subseteq \mathbf{V}$ is a subspace if and only if $U = \mathit{Ker}(f)$ for some linear function $f : \mathbf{V} \to \mathbf{F}^n$.

Proof.

$\Leftarrow$ We proved that $\mathrm{Ker}(f)$ is a subspace before.

Subspaces and kernels

Let $\mathbf{V}$ be a vector space over field $\mathbf{F}$.

Lemma

A set $U \subseteq \mathbf{V}$ is a subspace if and only if $U = Ker(f)$ for some linear function $f : \mathbf{V} \to \mathbf{F}^n$.

Proof.

$\Rightarrow$ Let $k = \dim(U)$, $m = \dim(\mathbf{V})$ and $n = m - k$. Let $u_1, \ldots, u_k$ be a basis of U. Extend it to basis $u_1, \ldots, u_m$ of $\mathbf{V}$. We define f by specifying its values on the basis:

$$f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq k \\ e_{i-k} & \text{for } k+1 \leq i \leq m \end{cases}$$

- $U \subseteq \mathrm{Ker}(f)$
- $\{e_1, \ldots, e_n\} \in \mathrm{Im}(f)$, hence $\dim(\mathrm{Im}(f)) = n$
- $\dim(\mathrm{Ker}(f)) = m - \dim(\mathrm{Im}(f)) = m - n = \dim(U)$, and thus $U = \mathrm{Ker}(f)$.

$\square$

Affine sets as solution sets

Corollary

A set $S \subseteq \mathbf{F}^m$ is a subspace if and only if it is the set of solutions of some system $Ax = 0$.

Corollary

A set $S \subseteq \mathbf{F}^m$ is affine if and only if it is the set of solutions of some system $Ax = b$.

Example

Problem

Find the equation of the plane $\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t \in \mathbf{R}\}$ *in* $\mathbf{R}^3$.

Example

Problem

Find the equation of the plane $\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t\in\mathbf{R}\}$ *in* $\mathbf{R}^3$*.*

- $(1,1,0),(1,2,3)$ is a basis of $U=\mathrm{span}(((1,1,0),(1,2,3))$.
- $B=(1,1,0),(1,2,3),(1,0,0)$ is a basis of $\mathbf{R}^3$.
- Let $f(1,1,0)=f(1,2,3)=(0)$, $f(1,0,0)=(1)$.
- We have $\mathrm{Ker}(f)=U$.
- $[f]_{B,D}=(0,0,1)$.

Example

Problem

Find the equation of the plane $\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t \in \mathbf{R}\}$ *in* $\mathbf{R}^3$*.*

- $B = (1,1,0),(1,2,3),(1,0,0)$
- Let $C = (1,0,0),(0,1,0),(0,0,1)$ be the standard basis of $\mathbf{R}^3$, $D = (1)$ the standard basis of $\mathbf{R}^1$.

$$
\begin{aligned}
[f]_{C,D} = [f]_{B,D}[\mathrm{id}]_{C,B} &= [f]_{B,D}[\mathrm{id}]_{B,C}^{-1} \\
&= (0,0,1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix}^{-1} = (1,-1,1/3)
\end{aligned}
$$

- Hence, $\mathrm{span}(((1,1,0),(1,2,3))$ is the set of solutions to $(1,-1,1/3)v = 0$.

Example

Problem

Find the equation of the plane $\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t \in \mathbf{R}\}$ *in* $\mathbf{R}^3$.

- $\text{span}((1,1,0),(1,2,3))$ is the set of solutions to $x-y+z/3=0$.
- For $(x,y,z)=(1,1,2)$, we have $x-y+z/3=2/3$.

$$\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t \in \mathbf{R}\}$$

is the set of solutions to

$$x-y+z/3=2/3.$$

Example

Problem

Find the equation of the plane $\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t \in \mathbf{R}\}$ *in* $\mathbf{R}^3$.

- $\mathrm{span}(((1,1,0),(1,2,3))$ is the set of solutions to $x - y + z/3 = 0$.
- For $(x,y,z) = (1,1,2)$, we have $x - y + z/3 = 2/3$.

$$\{(1,1,2)+(1,1,0)s+(1,2,3)t : s,t \in \mathbf{R}\}$$

is the set of solutions to

$$x - y + z/3 = 2/3.$$

Faster way: Find coefficients A, B, C, D such that $Ax+By+Cz = D$ is true for $(1,1,2)$, $(1,1,2)+(1,1,0)$, $(1,1,2)+(1,2,3)$.

Affine functions

Let $\mathbf{U}, \mathbf{V}$ be vector spaces over field $\mathbf{F}$.

Definition

A function $f : \mathbf{U} \to \mathbf{V}$ is affine if for every $u_1, \ldots, u_k \in \mathbf{U}$ and $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 + \ldots + \alpha_n = 1$, we have

$$f(\alpha_1 u_1 + \ldots + \alpha_k u_k) = \alpha_1 f(u_1) + \ldots + \alpha_k f(u_k).$$

- Every linear function is affine.
- The translation $f(x) = x + a$ is affine.
- Composition of affine functions is affine.

Affine functions as shifted linear functions

Lemma

For a function $f : \mathbf{U} \to \mathbf{V}$, the following claims are equivalent.

- *f is affine.*
- *The function $g : \mathbf{U} \to \mathbf{V}$, $g(x) = f(x) - f(o)$ is linear.*
- *There exists a linear function $g : \mathbf{U} \to \mathbf{V}$ and $a \in \mathbf{V}$ such that $f(x) = g(x) + a$ for every $x \in \mathbf{U}$.*

Proof.

1 $\Rightarrow$ 2 For every $x, y \in \mathbf{V}$ and $\alpha \in \mathbf{F}$, we have

$$\begin{aligned} g(x+y) &= f(x+y-o) - f(o) = (f(x) + f(y) - f(o)) - f(o) \\ &= g(x) + g(y) \\ g(\alpha x) &= f(\alpha x + (1-\alpha)o) - f(o) \\ &= (\alpha f(x) + (1-\alpha) f(o)) - f(o) = \alpha(f(x) - f(o)) \\ &= \alpha g(x) \end{aligned}$$

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- *There exists a linear function $g : \mathbf{U} \to \mathbf{V}$ and $a \in \mathbf{V}$ such that $f(x) = g(x) + a$ for every $x \in \mathbf{U}$.*

Proof.

2 $\Rightarrow$ 3 Set $a = f(o)$.

Affine functions as shifted linear functions

Lemma

For a function $f : \mathbf{U} \to \mathbf{V}$, the following claims are equivalent.

- *f is affine.*
- *The function $g : \mathbf{U} \to \mathbf{V}$, $g(x) = f(x) - f(o)$ is linear.*
- *There exists a linear function $g : \mathbf{U} \to \mathbf{V}$ and $a \in \mathbf{V}$ such that $f(x) = g(x) + a$ for every $x \in \mathbf{U}$.*

Proof.

3 $\Rightarrow$ 1 Suppose $\alpha_1 + \ldots + \alpha_k = 1$.

$$\begin{aligned} f(\alpha_1 v_1 + \ldots + \alpha_k v_k) &= g(\alpha_1 v_1 + \ldots + \alpha_k v_k) + a \\ &= \alpha_1 g(v_1) + \ldots + \alpha_k g(v_k) \\ &\quad + (\alpha_1 + \ldots + \alpha_k) a \\ &= \alpha_1 f(v_1) + \ldots + \alpha_k f(v_k) \end{aligned}$$

$\square$

Affine sets and functions

Lemma

For any affine function $f : \mathbf{U} \to \mathbf{V}$,

- *the set $Im(f) = \{f(u) : u \in \mathbf{U}\}$ is affine, and*
- *for every $v \in \mathbf{V}$, the set $f^{-1}(v) = \{u \in \mathbf{U} : f(u) = v\}$ is affine.*

Proof.

Let $f(x) = g(x) + a$ for linear function g.

$$\mathrm{Im}(f) = a + \mathrm{Im}(g).$$

$\square$

Affine sets and functions

Lemma

For any affine function $f : \mathbf{U} \to \mathbf{V}$*,*

- *the set* $Im(f) = \{f(u) : u \in \mathbf{U}\}$ *is affine, and*
- *for every* $v \in \mathbf{V}$*, the set* $f^{-1}(v) = \{u \in \mathbf{U} : f(u) = v\}$ *is affine.*

Proof.

Let $f(x) = g(x) + a$ for linear function g.

If $f^{-1}(v)$ is non-empty, choose $u_0 \in f^{-1}(v)$.

- $u \in f^{-1}(v)$ iff $o = f(u) - f(u_0) = g(u) - g(u_0) = g(u - u_0)$
  - I.e., $u - u_0 \in \text{Ker}(g)$.
- $f^{-1}(v) = u_0 + \text{Ker}(g)$.

$\square$

Computations with affine functions

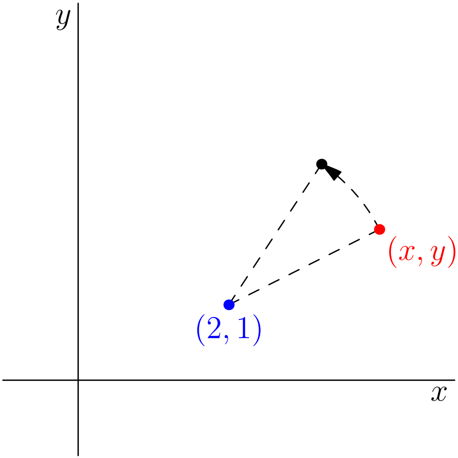
Since affine functions are just shifted linear functions $(f(x) = g(x) + a)$, we can:

- Describe f by coordinates of a and the matrix $[g]$.
- Evaluate f in coordinates.

Example

Problem

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation of the plane by 30 degrees around the point $(2, 1)$. To which point is (x, y) mapped by f?



Example

Problem

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation of the plane by 30 degrees around the point $(2, 1)$. To which point is (x, y) mapped by f?

- Let r be the rotation by 30 degrees around the point $(0, 0)$.
- Let t be the translation by $(2, 1)$.
- $f = trt^{-1}$

$$
\begin{aligned}
[r(v)]^T &= [r][v]^T = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} [v]^T \\
[t(v)]^T &= [v]^T + (2, 1)^T \\
[t^{-1}(v)]^T &= [v]^T - (2, 1)^T \\
[f(v)] &= [r]([v]^T - (2, 1)^T) + (2, 1)^T = [r][v]^T + (I - [r])(2, 1)^T
\end{aligned}
$$

Example

Problem

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation of the plane by 30 degrees around the point $(2, 1)$. To which point is (x, y) mapped by f?

$$[r] = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$[f(v)] = [r][v]^T + (I - [r])(2, 1)^T = [r][v]^T + (5/2 - \sqrt{3}, -\sqrt{3}/2)$$

Hence, $g(x, y) = (\sqrt{3}x/2 - y/2 + 5/2 - \sqrt{3}, x/2 + \sqrt{3}y/2 - \sqrt{3}/2)$.

A trick

For linear function g and affine function $f(x) = g(x) + a$, we have

$$[f(x)]^T = [g][x]^T + [a]^T.$$

Instead of using a matrix $[g]$ and vector $[a]$, we can use extended matrix

$$[[f]] = \begin{pmatrix} [g] & [a]^T \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} [f(x)]^T \\ 1 \end{pmatrix} = \begin{pmatrix} [g][x]^T + [a]^T \\ 1 \end{pmatrix} = [[f]] \begin{pmatrix} [x]^T \\ 1 \end{pmatrix}$$

Example, again

Problem

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ *be the rotation of the plane by* 30 *degrees around the point* $(2,1)$. *To which point is* (x,y) *mapped by* f*?*

- Let r be the rotation by 30 degrees around the point $(0,0)$.
- Let t be the translation by $(2,1)$.
- $f = trt^{-1}$

$$[[r]] = \begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[[t]]] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Example, again

Problem

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation of the plane by 30 degrees around the point $(2, 1)$. To which point is (x, y) mapped by f?

$$[[f]] = [[t]][[r]][[t]]^{-1} = \begin{pmatrix} \sqrt{3}/2 & -1/2 & 5/2 - \sqrt{3} \\ 1/2 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, $g(x, y) = (\sqrt{3}x/2 - y/2 + 5/2 - \sqrt{3}, x/2 + \sqrt{3}y/2 - \sqrt{3}/2)$.