Tree decompositions

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Definition 1. *A* tree decomposition *of a graph G is a pair (T, β), where*

- *T is a tree (we refer to the vertices of T as* nodes*, to make clear they are different from the vertices of G),*
- *β is a function assigning to each node x of T a subset of $V(G)$, called the* bag *of x,*
- *for every vertex v of G, there exists a node $x \in V(T)$ such that $v \in \beta(x)$,*
- *for every edge uv of G, there exists a node $x \in V(T)$ such that $u, v \in \beta(x)$, and*
- *for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in \beta(x)\}$ of nodes whose bag contains v induces a connected subtree of T.*

Lemma 1. *Let (T, β) be a tree decomposition of a graph G and let H be a subgraph of G. If H is connected, then the set*

$$\{x \in V(T) : V(H) \cap \beta(x) \neq \emptyset\}$$

induces a connected subtree T_H of T.

Proof. By induction on $|V(H)|$. The claim is true when $|V(H)| = 1$ by the last condition from the definition of the tree decomposition. Suppose now that $|V(H)| > 1$. There exists a vertex $v \in V(H)$ such that the graph $H - v$ is connected; e.g., we can choose v as a leaf of any spanning tree of H. Let u be a neighbor of v in H. By the induction hypothesis, T_{H-v} is a connected subtree of T. The last condition from the definition of the tree decomposition states that T_v is a connected subtree. Moreover, $T_{H-v} \cap T_v \neq \emptyset$, since uv is an edge of G and consequently there exists a node $x \in V(T)$ such that $u, v \in \beta(x)$. It follows that the subgraph $T_H = T_{H-v} \cup T_v$ of T is connected. □

Lemma 2. *Let (T, β) be a tree decomposition of a graph G. For every clique K in G, there exists a node $x \in V(T)$ such that $K \subseteq \beta(x)$.*

Proof. By induction on $|K|$. If $|K| \leq 2$, then the claim holds by the definition of the tree decomposition. Hence, suppose that $K = \{v_1, \ldots, v_k\}$ for some $k \geq 3$. Let $K_1 = \{v_1, \ldots, v_{k-1}\}$ and $K_2 = \{v_2, \ldots, v_k\}$. By the induction hypothesis, there exist nodes $x_1, x_2 \in V(T)$ such that $K_1 \subseteq \beta(x_1)$ and $K_2 \subseteq \beta(x_2)$. Let x be the last node on the path from x_1 to x_2 in T such that $v_1 \in \beta(x)$. The last condition from the definition of the tree decomposition implies that $K_1 \cap K_2 = \{v_2, \ldots, v_{k-1}\} \subseteq \beta(x)$. If $v_k \notin \beta(x)$, then observe that T cannot contain any node y such that $v_1, v_k \in \beta(y)$. This contradicts the definition of a tree decomposition, since $v_1 v_k$ is an edge. It follows that $K \subseteq \beta(x)$. □

1 Treewidth

Definition 2. *The* width *of a tree decomposition (T, β) is* $\max\{|\beta(v)| : v \in V(T)\} - 1$. *The* treewidth $\mathrm{tw}(G)$ *of G is the minimum width of its tree decomposition.*

Observation 3. *A graph G has treewidth at most t if and only if G can be obtained by clique-sums from graphs with at most $t+1$ vertices.*

Lemma 4. *If H is a minor G, then* $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

Proof. Let (T, β) be a tree decomposition of G of width $t = \mathrm{tw}(G)$. Let φ be a model of H in G. Let β' be the function assigning subsets of $V(H)$ to nodes of T such that for each node $x \in V(T)$, a vertex $v \in V(H)$ belongs to the bag $\beta'(x) \subseteq V(H)$ if and only if $\varphi(v) \cap \beta(x) \neq \emptyset$. Then (T, β') is a tree decomposition of H of width at most t (the last condition from the definition of the tree decomposition follows from Lemma 1). □

Observation 5. $\mathrm{tw}(K_n) = n - 1$.

Lemma 6. *Let G be a graph. Then*

- $\mathrm{tw}(G) = 0$ *iff $E(G) = \emptyset$, i.e., K_2 is not a minor of G;*
- $\mathrm{tw}(G) \leq 1$ *iff G is a forest, i.e., K_3 is not a minor of G; and*
- $\mathrm{tw}(G) \leq 2$, *iff K_4 is not a minor of G.*

Proof. The first two claims are easy; for the last one, use Lemma 4 from the notes for the notes `lesson15-5.pdf`. □

Definition 3. *A* bramble $\mathcal{B}$ *in a graph* G *is a set of non-empty subsets of* $V(G)$ *such that for all (not necessary distinct) sets* $X, Y \in \mathcal{B}$*, the subgraph of* G *induced by* $X \cup Y$ *is connected (and in particular, every set* $X \in \mathcal{B}$ *induces a connected subgraph). The* order *of the bramble* $\mathcal{B}$ *is the size of the smallest subset* $Z \subseteq V(G)$ *such that* $X \cap Z \neq \emptyset$ *for every* $X \in \mathcal{B}$*.*

Lemma 7. *Let* (T, β) *be a tree decomposition of a graph* G*. For every bramble* $\mathcal{B}$ *in* G*, there exists a node* $x \in V(T)$ *such that the bag* $\beta(v)$ *intersects all sets of* $\mathcal{B}$*.*

Proof. For an edge $xy \in E(T)$, let $T_{x,y}$ be the component of $T - xy$ containing y.

Suppose for a contradiction that for every node $x \in V(T)$, there exists a set $X \in \mathcal{B}$ disjoint from $\beta(x)$. Since $G[X]$ is connected, $\{z : \beta(z) \cap X \neq \emptyset\}$ induces a connected subtree T_X of T by Lemma 1. Therefore, there exists a unique neighbor y of the node x in T such that $T_X \subseteq T_{x,y}$. Let $\pi(x) := y$ and $X(x) := X$.

Since T is a tree, there exists an edge $xy \in E(T)$ such that $\pi(x) = y$ and $\pi(y) = x$. Since the set $X(x) \cup X(y)$ induces a connected subgraph of G, there exist vertices $u \in X(x)$ and $v \in X(y)$ such that $u = v$ or $uv \in E(G)$. By the definition of the tree decomposition, it follows that there exists a node $z \in V(T)$ such that $u, v \in \beta(z)$, and thus the subtrees $T_{X(x)}$ and $T_{X(y)}$ intersect in z. However, by the definition of π we have $T_{X(x)} \subseteq T_{x,y}$ and $T_{X(y)} \subseteq T_{y,x}$, which is a contradiction. □

Corollary 8. *If* (T, β) *is a tree decomposition of a graph* G *and* $\mathcal{B}$ *is a bramble in* G *of order* k*, then the tree decomposition* (T, β) *has width at least* $k - 1$*.*

Remark: It is actually true (but not easy to prove) that $\text{tw}(G) + 1 =$ maximum order of a bramble in G.

The $n \times m$ *grid* is the graph with vertex set $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ such that two vertices (i_1, j_1) and (i_2, j_2) are adjacent iff $|i_2 - i_1| + |j_2 - j_1| = 1$.

Lemma 9. *The* $n \times n$ *grid has treewidth at least* n*.*

Proof. Let S be the last column of the grid. Let R be the last row except for its last vertex. Let R_i be the i-th row except for the last vertex, and let S_j be the j-th column except for the last vertex. Let $\mathcal{B} = \{R, S\} \cup \{R_i \cup S_j : 1 \leq i, j \leq n - 1\}$. Then $\mathcal{B}$ is a bramble of order n. □

Corollary 10. *There exist* K_5*-minor-free graphs of arbitrarily large treewidth.*

2 Brambles and minors

Lemma 11. *Let $\mathcal{B}$ be a bramble in a graph G. There exists a path P in G intersecting all sets in $\mathcal{B}$.*

Proof. Let us construct P by gradually extending it, always choosing it so that there exists a set $X \in \mathcal{B}$ intersecting P exactly in an end x of P (initially, P consists of a single vertex contained in a set of the bramble). If there exists a set $X' \in \mathcal{B}$ disjoint from P, we extend P by adding a shortest path from x to a vertex of X' inside $G[X \cup X']$. This ensures that the resulting path intersects X' only in its end, as required. We keep extending the path in this way until it intersects all sets in $\mathcal{B}$. □

Observation 12. *Let $\mathcal{B}$ be a bramble in a graph G of order k, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be subsets of $\mathcal{B}$ such that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are brambles and the sum of their orders is at least k.*

Lemma 13. *Let $\mathcal{B}$ be a bramble of order at least $2t$ in a graph G and let P be a path in G intersecting all sets in $\mathcal{B}$. Then there exist vertex-disjoint subpaths P_1 and P_2 of P such that G contains t pairwise vertex-disjoint paths from P_1 to P_2.*

Proof. Let P_1 be the shortest initial segment of P such that the bramble $\mathcal{B}_1 = \{B \in \mathcal{B} : B \cap V(P) \neq \emptyset\}$ has order at least t. Let v be the last vertex of P. The subbramble $\mathcal{B}'_1 = \{B \in \mathcal{B} : B \cap V(P - v) \neq \emptyset\}$ has order at most $t-1$, and $\mathcal{B}_1$ is obtained from $\mathcal{B}'_1$ by adding sets intersected by a single vertex v, and thus $\mathcal{B}_1$ has order exactly t. By Observation 12, the bramble $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$ has order at least t. Let $P_2 = P - V(P_1)$ and note that since P intersects all sets in $\mathcal{B}$, P_2 intersects all sets in $\mathcal{B}_2$.

Consider any set $X \subseteq V(G)$ separating P_1 from P_2; we claim that $|X| \geq t$. Indeed, if $|X| < t$, then there exist sets $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ disjoint from X and $G[B_1 \cup B_2]$ contains a path from P_1 to P_2. By Menger's theorem, the graph G contains t pairwise vertex-disjoint paths from P_1 to P_2. □

The *ladder* Z_k is the graph consisting of the paths $u_1 \ldots u_k$ and $v_1 \ldots v_k$ and edges u_iv_i for $i = 1, \ldots, k$. I.e., Z_k is the $2 \times k$ grid.

Lemma 14. *Let $\mathcal{B}$ be a bramble in a graph G. If $\mathcal{B}$ has order at least $2k^2$, then G contains Z_k as a topological minor.*

Proof. Lemmas 11 and 13 imply that there exist vertex-disjoint paths P_1 and P_2 and k^2 pairwise vertex-disjoint paths Q_1, ..., Q_{k^2} from P_1 to P_2, without loss of generality intersecting P_1 and P_2 only in their ends. Let $x_1, \ldots, x_{k^2}$

and $y_1, \ldots, y_{k^2}$ be the ends of $Q_1, \ldots, Q_{k^2}$ on P_1 and P_2, respectively, in order along the paths P_1 and P_2. Let π be the permutation such that for every i, the path Q_i has ends x_i and $y_{\pi(i)}$. The sequence $\pi(1), \ldots, \pi(k^2)$ contains an increasing or a decreasing subset of length k. Then P_1, P_2, and the paths corresponding to this subsequence form a subdivision of Z_k. $\square$