Spanning trees

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1 Spanning trees with bounded maximum degree

See Czech notes.

2 Edge-disjoint spanning trees

Let G be a graph and let F_1, $\ldots$, F_k be edge-disjoint spanning (i.e., with $V(F_i) = V(G)$) forests in G. We say that the k-tuple $\mathcal{F} = (F_1, \ldots, F_k)$ is a *k-jungle*. Let $\bigcup \mathcal{F} = \bigcup_{i=1}^k F_i$ and $|\mathcal{F}| = |E(\bigcup \mathcal{F})|$. For a subgraph $H \subseteq G$, we say that a k-jungle $\mathcal{F}' = (F'_1, \ldots, F'_k)$ is *H-similar* to $\mathcal{F}$ if for every $i \in \{1, \ldots, k\}$,

- $E(F_i) \setminus E(H) = E(F'_i) \setminus E(H)$ (i.e., F_i and F'_i differ only on H), and
- $F_i \cap H$ and $F'_i \cap H$ have the same components.

The subgraph H is *$\mathcal{F}$-free* if for every $e \in E(H)$, there exists a k-jungle $\mathcal{F}'$ H-similar to $\mathcal{F}$ such that $e \notin E(\bigcup \mathcal{F}')$. Note that if $E(H) \cap E(\bigcup \mathcal{F}) = \emptyset$, then H is $\mathcal{F}$-free.

Lemma 1. *Let G be a graph and $\mathcal{F} = (F_1, \ldots, F_k)$ a k-jungle in G such that $|\mathcal{F}|$ is maximum. Let H be a maximal connected $\mathcal{F}$-free subgraph of G (i.e., no connected proper supergraph of H is $\mathcal{F}$-free); then for $i = 1, \ldots, k$, the subgraph $H \cap F_i$ is connected.*

Proof. Suppose for a contradiction that say $H \cap F_1$ is not connected. Since H is connected, there exists an edge $e_0 = uv \in E(H)$ with ends in different components of $H \cap F_1$. Since $e_0 \in E(H)$ and H is $\mathcal{F}$-free, there exists a k-jungle $\mathcal{F}' = (F'_1, \ldots, F'_k)$ H-similar to $\mathcal{F}$ such that $e_0 \notin E(\bigcup \mathcal{F}')$. If u and

v were in different components of F_1', then $(F_1' + e_0, F_2', \ldots, F_k')$ would be a k-jungle, contradicting the maximality of $|\mathcal{F}|$. Hence, this is not the case, and F_1' contains a path P with ends u and v. Since u and v are in different components of $F_1 \cap H$, they are also in different components of $F_1' \cap H$, and thus $H \cup P$ is a proper supergraph of H. Moreover, $H \cup P$ is clearly connected. However, $H \cup P$ is $\mathcal{F}$-free:

- For $e \in E(H)$, since H is $\mathcal{F}$-free, there exists a k-jungle $\mathcal{F}''$ H-similar to $\mathcal{F}$ with $e \notin E(\bigcup \mathcal{F}'')$; observe that $\mathcal{F}''$ is also $(H \cup P)$-similar to $\mathcal{F}$.
- For $e \in E(P) \setminus E(H)$, observe that the k-jungle $(F_1' - e + e_0, F_2', \ldots, F_k')$ is $(H \cup P)$-similar to $\mathcal{F}$.

But then $H \cup P$ contradicts the maximality of H. □

Let $\mathcal{P}$ be a partition of $V(G)$. Let $e(\mathcal{P})$ denote the number of edges of G with ends in different parts of $\mathcal{P}$.

Theorem 2. *The graph G has k edge-disjoint spanning trees if and only if every partition $\mathcal{P}$ of $V(G)$ satisfies $e(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$.*

Proof. For any spanning tree T in G, the graph obtained from T by contracting the parts of $\mathcal{P}$ is connected, and thus it has at least $|\mathcal{P}| - 1$ edges. Hence, if G has at least k edge-disjoint spanning trees, then $e(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$.

For the converse implication, we proceed by induction on $|V(G)$. The theorem clearly holds for graphs with one vertex, and thus assume that $|V(G)| \geq 2$. Let $\mathcal{P}_0$ be the partition of $V(G)$ to single-vertex parts; then

$$|E(G)| \geq e(\mathcal{P}_0) \geq k(|\mathcal{P}_0| - 1)|) = k(|V(G)| - 1).$$

Let $\mathcal{F} = (F_1, \ldots, F_k)$ be a k-jungle in G with $|\mathcal{F}|$ maximum. If $|\mathcal{F}| = k(|V(G)| - 1)$, then F_1, …, F_k are edge-disjoint spanning trees of G.

Hence, suppose that $|\mathcal{F}| < k(|V(G)| - 1) \leq |E(G)|$, and thus there exists an edge $e \in E(G) \setminus \bigcup \mathcal{F}$. The subgraph of G formed by e and its ends is connected and $\mathcal{F}$-free. Let H be a maximal connected $\mathcal{F}$-free subgraph of G with $e \in E(H)$. By Lemma 1, H has k edge-disjoint spanning trees $F_1 \cap H$, …, $F_k \cap H$. Since $e \in E(H)$, we have $|V(H)| \geq 2$.

Let G' be the graph obtained from G by contracting H to a single vertex h (we delete loops, but preserve parallel edges). Every partition $\mathcal{P}'$ of $V(G')$ corresponds to a partition $\mathcal{P}$ of $V(G)$ to the same number of parts, obtained by replacing h by the vertices of $V(H)$. Clearly $e(\mathcal{P}') = e(\mathcal{P}) \geq k(|\mathcal{P}| - 1) = (|\mathcal{P}'| - 1)$. Therefore, G' satisfies the assumptions of the theorem, and thus by the induction hypothesis, G' has k edge-disjoint spanning trees. We can combine them with the k edge-disjoint spanning trees of H, giving us k edge-disjoint spanning trees of G. □

Corollary 3. *Every $2k$-edge-connected graph has at least k edge-disjoint spanning trees.*

Theorem 4. *A graph G is a union of at most k forests if and only if every set $U \subseteq V(G)$ satisfies $|E(G[U])| \leq k(|U| - 1)$.*

Proof. If G is the union of at most k forests, then so is every induced subgraph H of G, and thus $|E(H)| \leq k(|V(H)| - 1)$.

For the converse implication, let $\mathcal{F}$ be a k-jungle in G G with $|\mathcal{F}|$ maximum. If $G \neq \bigcup \mathcal{F}$, then let e be an edge in $E(G) \setminus E(\bigcup \mathcal{F})$. Let H be a maximal connected $\mathcal{F}$-free subgraph of G with $e \in E(H)$. Lemma 1 implies that H has k edge-disjoint spanning trees, none of which contains e. Therefore, we have $|E(H)| \geq k(|V(H)| - 1) + 1$, contradicting the assumptions for $U = V(H)$. □