Connectivity and linkedness

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1 Disjoint paths in well-connected graphs

Let G be a graph and let $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(G)$ be pairwise distinct vertices. Pairwise vertex-disjoint paths $P_1, \ldots, P_k$, such that for $i = 1, \ldots, k$, the path P_i joins s_i with t_i, form an *$\vec{s} - \vec{t}$-linkage.* A graph G is *k-linked* if for all pairwise distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(G)$, there exists an $\vec{s} - \vec{t}$-linkage. Let us recall Menger's theorem.

Theorem 1. *Let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be pairwise distinct vertices of a graph G. Suppose that for all $A, B \subseteq G$ such that $G = A \cup B$, $\{s_1, \ldots, s_k\} \subseteq V(A)$, and $\{t_1, \ldots, t_k\} \subseteq V(B)$, we have $|V(A) \cap V(B)| \geq k$. Then G contains pairwise vertex-disjoint paths $P_1, \ldots, P_k$, such that for $i = 1, \ldots, k$, the path P_i joins s_i with one of the vertices $t_1, \ldots, t_n$.*

Unlike k-linkedness, Menger's theorem does not allow us to prescribe the ends of the paths. Indeed, k-connectivity does not imply k-linkedness: even 5-connected planar graphs are not 2-linked.

Returning to a postponed topic from the last lecture, we will show the following claim.

Lemma 2. *For any integer $k \geq 1$, if G is $2k$-connected and contains K_{4k} as a minor, then G is k-linked.*

Instead of Lemma 2, we prove the following stronger claim.

Lemma 3. *Let $k \geq 1$ be an integer, let G be a graph, let $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V(G)$ be pairwise distinct, and let $H_1, \ldots, H_{4k}$ be pairwise vertex-disjoint non-null subgraphs of G satisfying the following conditions.*

(a) For $i = 1, \ldots, 4k$, the subgraph H_i either is connected or every connected component of H_i intersects S.

(b) For $1 \leq i < j \leq 4k$, either G contains an edge with one end in H_i and the other end in H_j, or both H_i and H_j intersect S.

(c) If $A, B \subseteq G$, $G = A \cup B$, $S \subseteq V(A)$ and there exists $m \in \{1, \ldots, 4k\}$ such that $H_m \subseteq B - V(A)$, then $|V(A) \cap V(B)| \geq 2k$.

Then G contains an $\vec{s} - \vec{t}$-linkage.

Lemma 3 implies Lemma 2: Let H_1, $\ldots$, H_{4k} be the connected subgraphs of G contracted in order to create the K_{4k} minor. Hence, assumptions (a) and (b) hold. The assumption (c) holds by $2k$-connectivity of G. Let us now prove Lemma 3.

Proof. We proceed by induction on $|V(G)|+|E(G)|$; in particular, we assume that Lemma 3 holds for all proper minors of G.

Suppose first that the condition (c) holds sharply for some nontrivial $A, B \subseteq G$, that is, $G = A \cup B$, $G \neq B$, $S \subseteq V(A)$, $|V(A) \cap V(B)| = 2k$, and there exists $m \in \{1, \ldots, 4k\}$ such that $H_m \subseteq B - V(A)$. Let $S' = V(A) \cap V(B)$ a $H'_j = H_j \cap B$ for $1 \leq j \leq 4k$. Menger's theorem and condition (c) implies that there exist pairwise vertex-disjoint paths $S_1, \ldots, S_k, T_1, \ldots, T_k \subset G$ such that each of them has one end in S and the other end in S'. We can assume that $s_j \in V(S_j)$ and $t_j \in V(T_j)$ for $j = 1, \ldots, k$. Let s'_j denote the end of S_j in S', and let t'_j denote the end of T_j in S'.

Since $S \subset V(A)$ and $H_m \subseteq B - V(A)$, it follows that H_m is disjoint with S and $H'_m = H_m$. For $j = 1, \ldots, 4k$ different from m, the condition (b) implies that G contains an edge e joining H_m with H_j. Since one end of e lies in $B - V(A)$, we have $e \in E(B)$, and thus $H'_j = H_j \cap B$ is non-null.

Let us now argue that B, S', and H'_1, $\ldots$, H'_{4k} satisfy the assumptions of Lemma 3.

(a) If H'_j contains a component C disjoint with S', then C is a component of H_j as well and C is disjoint with S. By the condition (a) for G, we conclude that H_j is connected. Hence, $H_j = C$, and thus $H'_j = C$ is connected.

(b) Suppose that say H'_j does not intersect S'. By the preceding argument, $H'_j = H_j$ is connected and H_j is disjoint with S. By the condition (b) for G, G contains an edge e with one end in H_j and the other end in H_i. Since $H_j \subseteq B \setminus V(A)$, we have $e \in E(B)$, and thus B contains an edge with one end in H'_j and the other end in H'_i.

(c) Suppose that $B = A' \cup B'$, $S' \subseteq V(A')$ and there exists $m' \in \{1, \ldots, 4k\}$ such that $H'_{m'} \subseteq B' - V(A')$. By the preceding argument, $H'_{m'} = H_{m'}$.

Note that $G = (A \cup A') \cup B'$ and $S \subseteq V(A \cup A')$, hence the condition (c) for G gives $|V(A \cup A') \cap V(B')| \geq 2k$. However, $V(A \cup A') \cap V(B') = V(A') \cap V(B')$, since $V(A) \cap V(B') \subseteq V(A) \cap V(B) = S \subseteq V(A')$.

Since $B \subsetneq G$, we can apply induction to B. Therefore, there exists an $\vec{s'} - \vec{t'}$-linkage $P'_1, \ldots, P'_k$ in B. We obtain an $\vec{s} - \vec{t}$-linkage in G by letting $P_j = S_j \cup P'_j \cup T_j$ for $j = 1, \ldots, k$.

Hence, we can assume a stronger version of (c):

($\star$) If $A \subseteq G$ and $B \subsetneq G$ satisfy $G = A \cup B$, $S \subseteq V(A)$ a $H_m \subseteq B - V(A)$ for some $m \in \{1, \ldots, 4k\}$, then $|V(A) \cap V(B)| \geq 2k + 1$.

Consider any edge $e \in E(G)$. If both ends of e belong to S, then $G - e$ satisfies the assumptions of Lemma 3.

(a) If removing e disconnects some component of H_i, then both resulting components contain a vertex of S.

(b) If e is an edge between H_i and H_j, then both H_i and H_j intersect S.

(c) Trivially follows from (c) for G.

By the induction hypothesis, $G - e$ contains an $\vec{s} - \vec{t}$-linkage, and so does G. Hence, we can assume that e has at least one end outside of S.

Suppose that the ends of e do not belong to distinct subgraphs H_i and H_j, that is, either both ends of e belong to $V(H_i)$ for some $i \in \{1, \ldots, 4k\}$, or at least one end belongs to $V(G) \setminus (V(H_1) \cup \ldots \cup V(H_{4k}))$. Consider the graph G/e with subgraphs $H'_j = H_j/e$ for $j = 1, \ldots, 4k$. Assumptions (a) and (b) trivially hold. Suppose that $G/e = A' \cup B'$, where $S \subseteq V(A')$ and $H'_m \subseteq B' - V(A')$ for some $m \in \{1, \ldots, 4k\}$. Let A and B be subgraphs of G such that $G = A \cup B$, $A' = A/e$ and $B' = B/e$. Clearly, $S \subseteq V(A)$ and $H_m \subseteq B - V(A)$. If $B = G$, then $B' = G/e$ and $|V(A') \cap V(B')| \geq |S| = 2k$. If $B \neq G$, then by ($\star$) we have $|V(A) \cap V(B)| \geq 2k + 1$, and thus $|V(A') \cap V(B')| \geq |V(A) \cap V(B)| - 1 \geq 2k$. Hence, we can apply induction to G/e and obtain an $\vec{s} - \vec{t}$-linkage in G/e. Decontracting the edge e results in an $\vec{s} - \vec{t}$-linkage in G.

Similarly, we can delete any vertices not contained in $S \cup V(H_1) \cup \ldots \cup V(H_{4k})$. Therefore, we can assume that S is an independent set, every edge of G joins vertices in some distinct subgraphs H_i and H_j, and $V(G) = S \cup V(H_1) \cup \ldots \cup V(H_{4k})$. For $i = 1, \ldots, 4k$, the set $V(H_i)$ is independent, and by the assumption (a), either $V(H_i) \subseteq S$, or $|V(H_i)| = 1$. The assumption (b) impies that $G - S$ is a clique.

Now, let us prove the following:

($\star\star$) For every $S' \subseteq S$, there exist at least $|S'|$ vertices of G with a neighbor in S'.

Indeed, consider any such set $S' \subseteq S$, let Y be the set of vertices of G with a neighbor in S', and let $A = G[S \cup Y]$ and $B = G - S'$. Then $G = A \cup B$ and $S \subseteq V(A)$. If there exists no $m \in \{1, \ldots, 4k\}$ such that $H_m \subseteq B \setminus V(A)$, then $V(G) = V(A)$. However, $|V(G)| \geq 4k$ and $|V(A)| = |S| + |Y| = 2k + |Y|$, and thus $|Y| \geq 2k \geq |S'|$ as required. If $H_m \subseteq B \setminus V(A)$ for some $m \in \{1, \ldots, 4k\}$, then the assumption (c) implies that $|V(A) \cap V(B)| \geq 2k$. However, $|V(A) \cap V(B)| = |(S \setminus S') \cup Y| = 2k - |S'| + |Y|$, and thus again we get $|Y| \geq |S'|$.

By Hall's theorem, ($\star\star$) implies that there exists a matching $M \subseteq G$ of size $2k$ such that every edge of M has exactly one end in S. The matching M together with the edges of the clique $G - S$ contains an $\vec{s} - \vec{t}$-linkage. $\square$

2 Exercises

1. ($\star\star\star$) Prove that every 4-connected non-planar graph is 2-linked.

2. ($\star\star$) A graph G is *edge k-linked* if for every pairwise-distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exists pairwise edge-disjoint paths in G joinining s_1 with t_1, s_2 with t_2, $\ldots$, and s_k with t_k. Show that if G is 4-edge-connected, then G is edge 2-linked.

3. ($\star$) Find an example of a 2-edge-connected graph that is not edge 2-linked.