Minors, topological minors and degrees

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1 Minors and average degree

By results of Mader, Kostochka, and Thomasson, there exists $c > 0$ such that every graph on n vertices with at least $ck\sqrt{\log k} \cdot n$ edges contains K_k as a minor (and this result is tight, since there exists $c' > 0$ such that a random graph on $c'k\sqrt{\log k}$ vertices with high probability does not contain K_k as a minor). We are going to prove a somewhat weaker bound.

Let us start with a technical lemma.

Lemma 1. *Let $d \geq 1$ be an integer and let G be a graph with at least $d|V(G)|$ edges. Let G' be a minor of G such that $|E(G')| \geq d|V(G')|$ and $|V(G')| + |E(G')|$ is minimal. Every edge of G' is contained in at least d triangles, and the minimum degree of G' is at least $d+1$ and at most $2d$.*

Proof. Suppose that an edge xy of G' is contained in t triangles. Contracting the edge xy decreases the number of vertices by 1 and the number of edges by $t+1$. By the minimality of G', we have $t+1 > d$, and thus $t \geq d$. Similarly, removing a vertex v of G' of degree k decreases the number of vertices by 1 and the number of edges by k, and by the minimality of G', we have $k > d$. Finally, the minimality of G' implies that $|E(G')| = d|V(G')|$, that is, the average degree of G' is $2d$, and thus the minimum degree of G' is at most $2d$. □

By considering the neighbors of a vertex of G' of the minimum degree, we obtain the following consequence.

Corollary 2. *Let $d \geq 1$ be an integer. If a graph G has at least $d|V(G)|$ edges, then there exists a minor H of G such that $|V(H)| \leq 2d$ and H has minimum degree at least d.*

Now, we form an auxiliary *d-non-similarity* graph F of H with $V(F) = V(H)$ and two vertices $u, v \in V(F)$ adjacent if they have less than $d/3$ common neighbors in H.

Lemma 3. *Let $d \geq 1$ be an integer. Let H be a graph with $|V(H)| \leq 2d$. If H has minimum degree at least d, then the d-non-similarity graph of H is triangle-free.*

Proof. Let F be the d-non-similarity graph of H, and suppose that $uv, uw \in E(F)$ for distinct $u, v, w \in V(F)$. Let S be the set of non-neighbors of u in H; since H has minimum degree at least d, it follows that $|S| \leq |V(H)| - d \leq d$. Since $uv \in E(F)$, v has less than $d/3$ common neighbors with u in H, and since v has degree at least d, it has more than $\frac{2}{3}d$ neighbors in S. Similarly, w has more than $\frac{2}{3}d$ neighbors in S. It follows that v and w have more than $2 \cdot \frac{2}{3}d - |S| \geq d/3$ common neighbors in S, and thus $vw \notin E(F)$. Therefore, uvw is not a triangle in F. □

Let us recall a basic result from Ramsey theory.

Lemma 4. *For any integer $t \geq 0$, if F is a triangle-free graph with at least t^2 vertices, then F contains an independent set of size at least t.*

Proof. If F has maximum degree at least t, then the neighborhood of a vertex of maximum degree forms an independent set of size at least t. Hence, assume that every vertex in F has degree at most $t-1$. Let S be a maximal independent set in F, and let X be the set of vertices of F that have a neighbor in S. Since S is maximal, we have $V(F) = S \cup X$, and thus $|S| + |X| \geq t^2$. However, since every vertex of F has degree at most $t-1$, we have $|X| \leq (t-1)|S|$, and thus $|S| + |X| \leq t|S|$. By comparing the inequalities, we conclude that $|S| \geq t$. □

We are now ready to prove the result on the density of graphs without K_k minor.

Theorem 5. *Let $k \geq 1$ be an integer, and let $d = \frac{3}{2}k(k+1)$. If a graph G has at least $d|V(G)|$ edges, then G contains K_k as a minor.*

Proof. Let H be the minor of G obtained using Corollary 2, such that $|V(H)| \leq 2d$ and H has minimum degree at least d. Let F be the d-non-similarity graph of H. By Lemma 3, F is triangle-free, and by Lemma 4, F contains an independent set S of size k.

Consider any two vertices $u, v \in S$. Since $uv \notin E(F)$, u and v have at least $d/3$ common neighbors in H, and at least $d/3 - k = \binom{k}{2}$ of them are not contained in S. Therefore, for every pair $\{u, v\} \subseteq S$, we can choose a vertex m_{uv} adjacent to both u and v and not belonging to S, such that the choices are pairwise distinct for different pairs of vertices of S. The union of the paths $um_{uv}v$ for $\{u, v\} \subseteq S$ forms a subdivision of K_k in H. Since $H \leq_m G$, we conclude that $K_k \leq_m G$. □

Let us remark that this proves a weak version of Hadwiger's conjecture.

Corollary 6. *For any integer $k \geq 1$, if G does not contain K_k as a minor, then $\chi(G) \leq 3k(k+1)$.*

2 Disjoint paths in connected graphs

Let G be a graph and let $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(G)$ be pairwise distinct vertices. Pairwise vertex-disjoint paths $P_1, \ldots, P_k$, such that for $i = 1, \ldots, k$, the path P_i joins s_i with t_i, form an *$\vec{s} - \vec{t}$-linkage.* A graph G is *k-linked* if for all pairwise distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(G)$, there exists an $\vec{s} - \vec{t}$-linkage. Let us recall Menger's theorem.

Theorem 7. *Let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be pairwise distinct vertices of a graph G. Suppose that for all $A, B \subseteq G$ such that $G = A \cup B$, $\{s_1, \ldots, s_k\} \subseteq V(A)$, and $\{t_1, \ldots, t_k\} \subseteq V(B)$, we have $|V(A) \cap V(B)| \geq k$. Then G contains pairwise vertex-disjoint paths $P_1, \ldots, P_k$, such that for $i = 1, \ldots, k$, the path P_i joins s_i with one of the vertices $t_1, \ldots, t_n$.*

Unlike k-linkedness, Menger's theorem does not allow us to prescribe the ends of the paths. Indeed, k-connectivity does not imply k-linkedness: even 5-connected planar graphs are not 2-linked. However, we can prove that sufficiently high connectivity implies k-linkedness.

Theorem 8. *For any integer $k \geq 1$, if G is $12k(4k+1)$-connected, then G is k-linked.*

Thomas and Wollan proved that actually $5k$-connectivity implies k-linkedness. Theorem 8 is a corollary of Theorem 5 and the following claim.

Lemma 9. *For any integer $k \geq 1$, if G is $2k$-connected and contains K_{4k} as a minor, then G is k-linked.*

We postpone the proof of Lemma 9 for the next lecture.

3 Topological minors and average degree

As an easy corollary of Theorem 8, we obtain the following result on the existence of topological minors.

Lemma 10. *For any $k \geq 1$, let $d = 12\binom{k}{2}\left(4\binom{k}{2}+1\right) + k = O(k^4)$. Every d-connected graph contains a subdivision of K_k.*

Proof. Let G be a d-connected graph, and let $v_1, \ldots, v_k$ be arbitrary vertices of G. Since the minimum degree of G is greater than $k(k-1)$, we can select pairwise distinct vertices v_{ij} for all $i, j \in \{1, \ldots, k\}$, $i \neq j$, so that v_{ij} is a neighbor of v_i. The graph $G - \{v_1, \ldots, v_k\}$ is $d - k = 12\binom{k}{2}\left(4\binom{k}{2}+1\right)$-connected, and by Theorem 8, it is $\binom{k}{2}$-linked. Hence, it contains pairwise disjoint paths P_{ij} joining v_{ij} with v_{ji} for $1 \leq i < j \leq k$. These paths together with the stars around $v_1, \ldots, v_n$ give a subdivision of K_k. □

We are going to use the following interesting result by Mader.

Lemma 11. *For every integer $d \geq 1$, if a graph G has at least $2d|V(G)|$ edges, then G contains a $(d+1)$-connected subgraph.*

Proof. Let H be a smallest subgraph of G such that $|V(H)| \geq 2d$ and $|E(H)| > 2d(|V(H)| - d)$. If $|V(H)| = 2d$, then $|E(H)| > 2d^2 > \binom{|V(H)|}{2}$, which is a contradiction. Therefore, $|V(H)| \geq 2d + 1$. By the minimality of H, removing each vertex results in removal of at least $2d$ edges, and thus H has minimum degree at least $2d$.

Consider any proper induced subgraphs A and B of H such that $H = A \cup B$. Any vertex in $V(A) \setminus V(B)$ has all its neighbors in A, and thus $|V(A)| > 2d$, and similarly $|V(B)| > 2d$. By the minimality of H, we have $|E(A)| \leq 2d(|V(A)| - d)$ and $|E(B)| \leq 2d(|V(B)| - d)$. Consequently,

$$\begin{aligned} |E(H)| &\leq |E(A)| + |E(B)| \\ &\leq 2d(|V(A)| + |V(B)| - 2d) \\ &= 2d(|V(H)| - d + |V(A) \cap V(B)| - d). \end{aligned}$$

Since $|E(H)| > 2d(|V(H)| - d)$, it follows that $|V(A) \cap V(B)| > d$, and thus G has no cut of size at most d. □

In conclusion, we have the following.

Corollary 12. *For any $k \geq 1$, let $d = 24\binom{k}{2}\left(4\binom{k}{2}+1\right) + 2k = O(k^4)$. Every graph G with at least $d|V(G)|$ edges contains a subdivision of K_k.*

By the results of Thomas and Wollan, $|E(G)| \geq 10k^2|V(G)|$ suffices to force the existence of a subdivision of K_k.

4 Exercises

1. ($\star\star$) Prove that every graph G with at least four vertices and at least $2|V(G)| - 2$ edges contains K_4 as a minor.

2. ($\star$) Prove that if a graph G is t-linked for every $t \leq k$, then G is k-connected.

3. ($\star\star$) Assume that the following claim is true: every 4-connected non-planar graph is 2-linked. Let G be a planar 4-connected graph and let s_1, s_2, t_1, and t_2 be pairwise distinct vertices of G. Prove that G contains an $\vec{s} - \vec{t}$-linkage unless G has a face bounded by a cycle C containing s_1, s_2, t_1, and t_2 such that $\{s_1, t_1\}$ separates s_2 from t_2 in C.

4. ($\star$) Show that there exist some constant $c > 0$ such that for every integer $k \geq 2$, there exists a graph G with at least $ck^2|V(G)|$ edges that does not contain a subdivision of K_k.