Characterizations of graph classes by forbidden configurations

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We consider graph classes that can be described by excluding some fixed configurations. Let us give some examples.

Theorem 1 (Kuratowski)**.** *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a topological minor (i.e., does not contain a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$).*

Theorem 2. *A graph G is bipartite if and only if it does not contain any odd cycle as a subgraph.*

Let us give a few definitions. We consider the following partial orders on graphs:

- *induced subgraph* $H \sqsubseteq G$
- *subgraph* $H \subseteq G$
- *topological minor* $H \le_t G$ (a subdivision of H is a subgraph of G)
- *minor* $H \le_m G$ (H can be obtained from a subgraph of G by contracting edges)

Note that $H \sqsubseteq G \Rightarrow H \subseteq G \Rightarrow H \le_t G \Rightarrow H \le_m G$.

Let $\preceq$ be any of these orders. We say that a class $\mathcal{G}$ of graphs is $\preceq$*-closed* (subgraph-closed, minor-closed, ...) if for all graphs H and G, if $H \preceq G$ and $G \in \mathcal{G}$, then $H \in \mathcal{G}$.

Examples:

- The class of all planar graphs is minor-closed (and thus it also is topological-minor-closed, subgraph-closed and induced-subgraph-closed).

- The class of all bipartite graphs is subgraph-closed, but not topological-minor-closed.
- The class of all graphs whose connected components are cliques is induced-subgraph-closed, but not subgraph-closed.

For a set $\mathcal{F}$ of graphs, let $\mathrm{Forb}_{\preceq}(\mathcal{F}) = \{G : (\forall F \in \mathcal{F}) F \not\preceq G\}$ denote the class of graphs that do not "contain" any element of $\mathcal{F}$ in the sense defined by $\preceq$.

Examples:

- $\mathrm{Forb}_{\leq_t}(K_5, K_{3,3})$ = planar graphs.
- $\mathrm{Forb}_{\subseteq}(C_3, C_5, C_7, \ldots)$ = bipartite graphs.
- $\mathrm{Forb}_{\sqsubseteq}(K_{1,2})$ = graphs whose connected components are cliques.

A graph F is a $\preceq$-*obstruction* for a graph class $\mathcal{G}$ if $F \notin \mathcal{G}$, but $F' \in \mathcal{G}$ for all $F' \prec F$. Let $\mathrm{Obst}_{\preceq}(\mathcal{G})$ be the set of all $\preceq$-obstructions for $\mathcal{G}$.

Examples:

- K_5 is a topological-minor-obstruction for planar graphs, since K_5 is not planar, but all proper topological minors of K_5 are planar.
- K_6 is not a topological-minor-obstruction for planar graphs, since $K_5 <_t K_6$ and K_5 is not planar.

We say that a partial order $\preceq$ on graphs is *locally finite* if $\{H : H \preceq G\}$ is finite for every graph G.

Lemma 3. *Let $\mathcal{G}$ be a class of graphs, and let $\preceq$ be a locally finite order. The following claims are equivalent.*

(a) $\mathcal{G}$ is $\preceq$-closed

(b) $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathcal{F})$ for some set $\mathcal{F}$

(c) $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathrm{Obst}_{\preceq}(\mathcal{G}))$

Proof.

(a) ⇒ (c) First, suppose that $G \in \mathcal{G}$. Since $\mathcal{G}$ is $\preceq$-closed, every $H \preceq G$ satisfies $H \in \mathcal{G}$, and thus H is not a $\preceq$-obstruction for $\mathcal{G}$. Hence, $G \in \mathrm{Forb}_{\preceq}(\mathrm{Obst}_{\preceq}(\mathcal{G}))$.

Consider now any graph $G \notin \mathcal{G}$. The set $S = \{H \preceq G : H \notin \mathcal{G}\}$ is finite, and thus it contains a $\preceq$-minimal element F. Then $F \notin \mathcal{G}$, but $F' \in \mathcal{G}$ for every $F' \prec F$, i.e., F is a $\preceq$-obstruction for $\mathcal{G}$. Hence $G \notin \mathrm{Forb}_{\preceq}(\mathrm{Obst}_{\preceq}(\mathcal{G}))$.

Therefore, $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathrm{Obst}_{\preceq}(\mathcal{G}))$.

(c) ⇒ (b) Trivial, let $\mathcal{F} = \mathrm{Obst}_{\preceq}(\mathcal{G})$.

(b) ⇒ (a) Consider any $G \in \mathcal{G}$. Since $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathcal{F})$, we have $F \not\preceq G$ for every $F \in \mathcal{F}$. Consequently, if $H \preceq G$, then also $F \not\preceq H$. Therefore, $H \in \mathrm{Forb}_{\preceq}(\mathcal{F}) = \mathcal{G}$. Since this holds for every $G \in \mathcal{G}$ and every $H \preceq G$, the class $\mathcal{G}$ is $\preceq$-closed.

□

1 Subgraph-closed classes

Let P_n denote a path with n vertices, and let tK_2 denote the matching of size t. Simple examples:

- $\mathrm{Forb}_{\subseteq}(C_3, C_4, C_5, \ldots)$ = forests
- $\mathrm{Forb}_{\subseteq}(C_3, C_5, C_7, \ldots)$ = bipartite graphs
- $\mathrm{Forb}_{\subseteq}(P_2)$ = isolated vertices
- $\mathrm{Forb}_{\subseteq}(P_3)$ = isolated vertices and edges
- $\mathrm{Forb}_{\subseteq}(K_{1,n})$ = maximum degree at most $n-1$
- $\mathrm{Forb}_{\subseteq}(2K_2)$ = isolated vertices, or a star plus isolated vertices, or a triangle plus isolated vertices.

$\mathrm{Forb}_{\subseteq}(tK_2)$ is the class of graphs with maximum matching of size at most $t-1$, which can be described explicitly using Tutte's theorem. The following approximate description is often more useful. A set $X \subseteq V(G)$ is a *vertex cover* if every edge of G is incident with a vertex of X, i.e., $G - X$ has no edges.

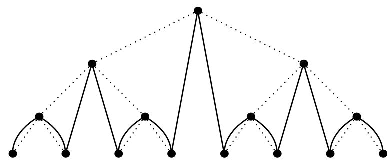
Theorem 4. *Every graph in* $\mathrm{Forb}_{\subseteq}(tK_2)$ *has a vertex cover of size at most* $2(t-1)$. *Conversely, every graph with vertex cover of size at most t belongs to* $\mathrm{Forb}_{\subseteq}((t+1)K_2)$.

Proof. Suppose that $tK_2 \not\subseteq G$. Let $M \subseteq G$ be a maximum matching, $|E(M)| \le t-1$. Then $G - V(M)$ has no edges, i.e., $V(M)$ is a vertex cover for G of size at most $2(t-1)$.

Conversely, if X is a vertex-cover of G of size at most t, then every edge of a matching in G intersects X, and thus such a matching has at most $|X|$ edges. Consequently, $(t+1)K_2 \not\subseteq G$. □

We can also obtain a similar approximate characterization for $\mathrm{Forb}_{\subseteq}(P_n)$. The *tree-depth* of a graph G is the smallest $d \ge 1$ for that there exists a rooted forest T of depth at most d with vertex set $V(G)$, such that every edge of G joins a vertex with its ancestor or descendant in T. Examples:

- Graphs of tree-depth 1 consist of isolated vertices.
- Graphs of tree-depth at most 2 consist of stars.
- The path P_{2^n-1} has tree-depth n, the path P_{2^n} has tree-depth $n+1$.



Theorem 5. *If $P_n \not\subseteq G$, then G has tree-depth at most $n-1$. Conversely, if G has tree-depth at most n, then $P_{2^n} \not\subseteq G$.*

Proof. Suppose that $P_n \not\subseteq G$. We can assume that G is connected, as otherwise we consider each component separately. Run depth-first search from any vertex of G, and let T be the resulting tree. Then $T \subseteq G$, hence $P_n \not\subseteq T$, and thus T has depth at most $n-1$. Observe also that every edge of G joins a vertex with its ancestor or descendant in T.

Conversely, if $P_{2^n} \subseteq G$, then G has at least as large tree-depth as P_{2^n}, which is $n+1$. □

2 Induced-subgraph-closed classes

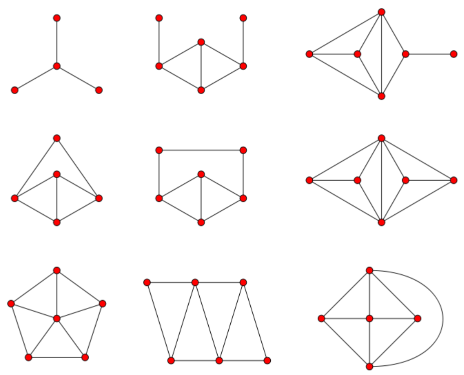
For a graph H, let $\overline{H}$ denote the complement of H, that is the graph with the same vertex set in that two distinct vertices are adjacent if and only if they are not adjacent in H.

Let us mention a famous recent result. A graph G is *perfect* if $\omega(H) = \chi(H)$ for every $H \sqsubseteq G$. Perfect graphs are interesting, since we can determine their chromatic number as well as the size of maximum clique in polynomial time. A *hole* is a cycle of length at least 4. An *anti-hole* is a complement of a hole. The following characterization of perfect graphs was proposed by Berge in 1961, and finally proved by Chudnovsky, Robertson, Seymour, and Thomas in 2002.

Theorem 6. *A graph is perfect if and only if it contains neither odd hole nor an odd anti-hole as an induced subgraph. That is,* $\mathrm{Forb}_{\sqsubseteq}(C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots) =$ *perfect graphs.*

Another well-known result concerns line-graphs. A graph G is a *line-graph* of H if $V(G) = E(H)$, and two vertices of G are adjacent if and only if the corresponding edges of H are incident with the same vertex.

Theorem 7. *A graph G is a line-graph of some graph if and only if it does not contain any of the following graphs as an induced subgraph:*



Further examples:

- $\mathrm{Forb}_{\sqsubseteq}(C_3, C_4, C_5, \ldots) =$ forests
- $\mathrm{Forb}_{\sqsubseteq}(C_3, C_5, C_7, \ldots) =$ bipartite graphs
- $\mathrm{Forb}_{\sqsubseteq}(P_2) =$ isolated vertices
- $\mathrm{Forb}_{\sqsubseteq}(P_3) =$ all components are cliques
- Because $\overline{C_4}$ is equal to $2K_2$, $\mathrm{Forb}_{\sqsubseteq}(2K_2)$ contains exactly the complements of graphs in $\mathrm{Forb}_{\sqsubseteq}(C_4)$, and in particular complements of all graphs without cycles of length at most 4. The exact description of $\mathrm{Forb}_{\sqsubseteq}(2K_2)$ is not known. See homework exercises for some partial results.
- The description of $\mathrm{Forb}_{\sqsubseteq}(K_{1,3})$ (*claw-free graphs*) is known, but it is extremely complicated.

3 Exercises

1. ($\star$) Let $\mathcal{G}$ be a $\preceq$-closed class of graphs, where $\preceq$ is a locally finite order. Show that $\mathrm{Obst}_{\preceq}(\mathcal{G}) \subseteq \mathcal{F}$ for every set $\mathcal{F}$ such that $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathcal{F})$.

2. ($\star$) Describe the graphs in $\mathrm{Forb}_{\subseteq}(P_4)$.

3. ($\star\star$) Prove that P_{2^n} has tree-depth $n+1$.

4. ($\star$) Prove that $\mathrm{Forb}_{\sqsubseteq}(C_3, C_5, C_7, \ldots) =$ bipartite.

5. ($\star\star\star$) Describe the graphs in $\mathrm{Forb}_{\sqsubseteq}(2K_2, C_3, C_5, C_7, \ldots)$, that is bipartite graphs without induced matching of size 2.