1. Let $v_1, \ldots, v_n$ be an ordering of vertices of a graph G such that $v_1v_2 \in E(G)$ and for every $i \geq 3$, the vertex v_i has exactly two neighbors in $\{v_1, \ldots, v_{i-1}\}$. Let

$$p_G = \prod_{v_iv_j \in E(G), i<j} (x_j - x_i)$$

be the graph polynomial of G. For any function $f : [n] \to \mathbb{N}$, let us define $c(f, G)$ as the coefficient at $x_1^{2-f(1)} x_2^{2-f(2)} \ldots x_n^{2-f(n)}$ in the polynomial p_G (if $f(i) > 2$ for some i, then $c(f, G) = 0$). Let $e_i : [n] \to \mathbb{N}$ be the function such that $e_i(i) = 1$ and $e_i(j) = 0$ pro $j \neq i$. Let us define $\widetilde{c}(f, G) = c(f + e_2, G) - c(f + e_1, G)$.

   Prove by induction that for $n \geq 2$, if f satisfies $\sum_{i=1}^{n} f(i) = 2$, then $\widetilde{c}(f, G) \equiv 1 \pmod 3$.

2. Let G be a 2-degenerate graph, let z be a vertex of G and let L be a list assignment for G such that $|L(v)| \geq 3$ for every $v \in V(G) \setminus \{z\}$ and $|L(z)| = 1$. Using the previous exercise, prove that G is colorable from the lists L.

3. Suppose p is a prime. Prove that if G is a multigraph of average degree greater than $2p - 2$, then G has a non-empty submultigraph in which the degrees of all vertices are divisible by p.

4. We say that a graph G is *almost d-regular* if all its vertices have degree d or $d+1$. Prove that if G is almost d-regular and the vertices of degree $d+1$ form an independent set, then G contains a matching that covers all vertices of degree $d + 1$.

5. Using the previous exercise, prove that if G is almost d-regular, then G has a spanning subgraph that is almost $(d - 1)$-regular, but not $(d - 1)$-regular.

6. Using the previous exercise, prove that if G is almost d-regular for $d \geq 4$ and G is not 4-regular, then G has a 3-regular subgraph.

7. Let p and p_1 be polynomials in variables $x_1, \ldots, x_n$. Let L be a finite set of complex numbers and let q be the polynomial $\prod_{r \in L}(x_1 - r)$, and let a be the polynomial $p - p_1 q$. Show that $a(x_1, \ldots, x_n) = p(x_1, \ldots, x_n)$ holds for all complex numbers $x_1, \ldots, x_n$ such that $x_1 \in L$.

8. Let p be a polynomial in variables $x_1, \ldots, x_n$. For $i = 1, \ldots, n$, let L_i be a finite set of complex numbers and let $d_i = |L_i| - 1$. Suppose

that the degree of p is $d_1 + \ldots + d_n$. Using the previous exercise, prove there exists a polynomial a in variables $x_1, \ldots, x_n$ such that $a(x_1, \ldots, x_n) = p(x_1, \ldots, x_n)$ for every $x_1 \in L_1, \ldots, x_n \in L_n$, every variable x_i appears in a in degree at most d_i, and a and p have the same coefficient at $x_1^{d_1} \cdots x_n^{d_n}$.

9. Let a be a non-zero polynomial in variables $x_1, \ldots, x_n$. For $i = 1, \ldots, n$, let L_i be a finite set of complex numbers and let $d_i = |L_i| - 1$. Suppose every variable x_i appears in a in degree at most d_i. Prove there exist $x_1 \in L_1, \ldots, x_n \in L_n$ such that $a(x_1, \ldots, x_n) \neq 0$. (Hint: use the fact that every non-zero polynomial of degree d in one variable has at most d roots).

10. Let p be a polynomial in variables $x_1, \ldots, x_n$. For $i = 1, \ldots, n$, let L_i be a finite set of complex numbers and let $d_i = |L_i| - 1$. Suppose that the degree of p is $d_1 + \ldots + d_n$ and the coefficient of p at $x_1^{d_1} \cdots x_n^{d_n}$ is non-zero. Using the previous exercises, prove that there exist $x_1 \in L_1, \ldots, x_n \in L_n$ such that $p(x_1, \ldots, x_n) \neq 0$.

11. Use the previous exercises to prove the Alon-Tarsi theorem on list-coloring.