1. Determine the treewidth of the cube and the octahedron graphs.

2. Prove that the treewidth of the $n \times n$ grid is at most n.

3. Prove that every graph G has a tree decomposition (T, β) of width $\mathrm{tw}(G)$ such that $|V(T)| \leq |V(G)|$.

4. A set W of $2k+1$ vertices of G is *k-breakable* if there exists $X \subseteq V(G)$ of size at most k such that each component of $G - X$ contains at most k vertices of W. Prove that if every set $W \subseteq V(G)$ of size $2k+1$ is k-breakable, then $\mathrm{tw}(G) \leq 3k$ (hint: prove by induction on the number of vertices of H that for every $H \subseteq G$ and for every $W \subseteq V(H)$ of size at most $2k+1$, there exists a tree decomposition (T_H, β_H) of H of width at most $3k$ such that $W \subseteq \beta_H(v)$ for some $v \in V(T_H)$).

5. Let W be a set of $2k+1$ vertices in a graph G. Let $\mathcal{B} = \{X \subseteq V(G) : G[X]$ is connected and $|X \cap W| \geq k+1\}$. Prove that $\mathcal{B}$ is a bramble in G. Furthermore, show that if W is not k-breakable, then $\mathcal{B}$ has order at least $k+1$.

6. Prove that if $\mathrm{tw}(G) > 3k$, then G contains a bramble of order at least $k+1$.