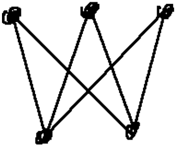
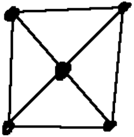
Definition

A Hamiltonian cycle in G is a cycle $C \subseteq G$ such that $V(C) = V(G)$. A graph is Hamiltonian if it has a Hamiltonian cycle.

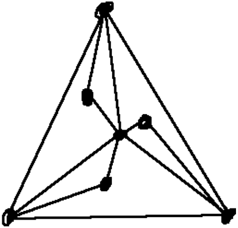
Q: Which of the following graphs are Hamiltonian?



G_1



G_2



G_3

Observation

If G is Hamiltonian, then for every $S \subseteq V(G)$, $G - S$ has at most $|S|$ components.

A graph G is t-tough if for every $S \subseteq V(G)$, $G - S$ has at most $\max(1, |S|/t)$ components.

Observation

Every Hamiltonian graph is 1-tough.

Conjecture

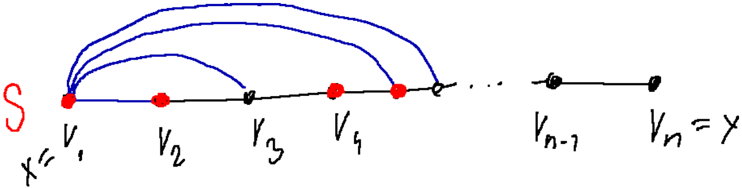
There exists c such that every c-tough graph is Hamiltonian.

We know c (if it exists) must be at least $9/4$.

Lemma

Suppose x, y *are non-adjacent and* $\deg x + \deg y \geq |V(G)|$. *If* $G + xy$ *is Hamiltonian, then* G *is Hamiltonian.*

- $v_1 v_2 \ldots v_n =$ Hamiltonian cycle minus $xy = v_1 v_n$
- $S = \{v_i : v_1 v_{i+1} \in E(G)\} \subseteq \{v_1, v_2, \ldots, v_{n-1}\}$
- $|S| + \deg y = \deg x + \deg y \geq n \Rightarrow N(y) \cap S \neq \emptyset$.



Chvátal closure:

- as long as there exist non-adjacent x, y s.t. $\deg x + \deg y \geq |V(G)|$, add the edge xy.

Corollary

A graph is Hamiltonian iff its Chvátal closure is Hamiltonian.

Corollary (Ore's theorem)

If $|V(G)| \geq 3$ *and* $\deg x + \deg y \geq |V(G)|$ *holds for all non-adjacent* $x, y \in V(G)$*, then* G *is Hamiltonian.*

Corollary (Dirac's theorem)

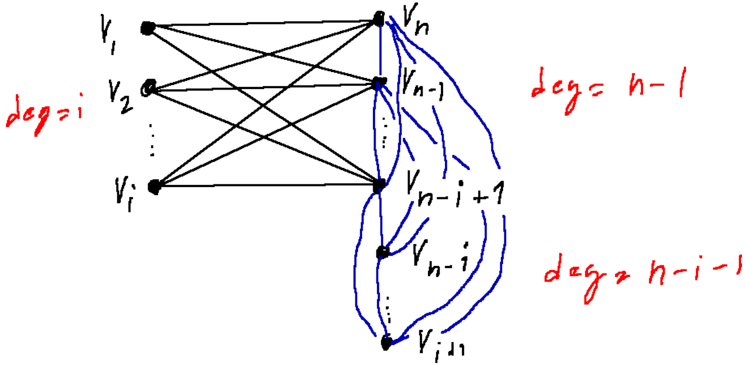
If $|V(G)| \geq 3$ *and* $\delta(G) \geq |V(G)|/2$*, then* G *is Hamiltonian.*

Q: For every n, find a non-Hamiltonian graph G with n vertices and $\delta(G) \geq n/2 - 1$.

The best theorem of form

"deg $v_1 \geq a_1$, deg $v_2 \geq a_2, \ldots,$ deg $v_n \geq a_n$ implies G is Hamiltonian"?

- WLOG $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n - 1$.
- For $i < n/2$: If $a_i \leq i$, we cannot have $a_{n-i} < n - i$:



Theorem

Suppose that $n \geq 3$ *and*

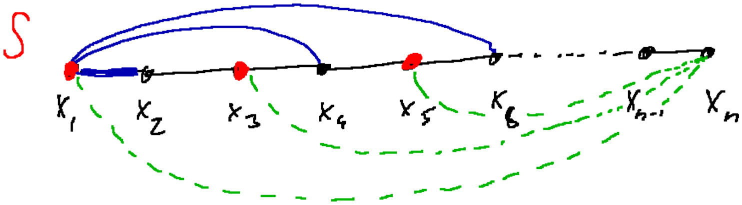
- $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n-1$.
- *For* $i = 1, \ldots, \lfloor (n-1)/2 \rfloor$, *if* $a_i \leq i$, *then* $a_{n-i} \geq n-i$.

If G is a graph whose vertices $v_1, \ldots, v_n$ *satisfy* $\deg v_i \geq a_i$, *then G is Hamiltonian.*

WLOG:

- $\deg x + \deg y \leq n-1$ for all non-adjacent x, y
- $\deg v_1 \leq \deg v_2 \leq \ldots \leq \deg v_n$
- $G +$ any edge is Hamiltonian.

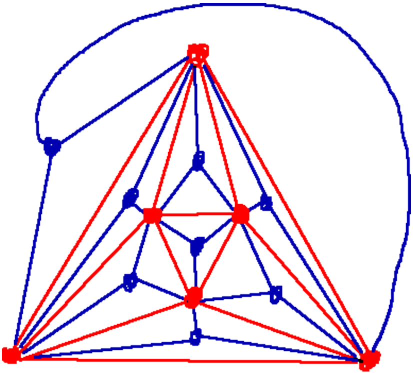
- x_1, x_n non-adjacent, $\deg x_1 + \deg x_n$ maximum, $\deg x_1 \le \deg x_n$.
- $x_1 x_2 \ldots x_n$ path in G
- $S = \{x_i : x_1 x_{i+1} \in E(G)\}$: $N(x_n) \cap S = \emptyset$



$h = \deg x_1 = |S| < n/2$

- $\deg v \le h$ for $v \in S \Rightarrow a_h \le h$
- x_1 has a non-neighbor $z \in \{v_{n-h}, \ldots, v_n\}$.
- $a_{n-h} \ge n - h \Rightarrow \deg x_1 + \deg z \ge n$ ↯

There exist non-Hamiltonian plane triangulations:



$G - S$ has $2|S| - 4 > |S|$ components.

Theorem (Tutte)

Every 4-connected planar graph is Hamiltonian.

The attachments of a component C of $G - S$ are vertices in S with neighbors in C.

Lemma

Let G be a 2-connected plane graph, x, y vertices incident with the outer face. There exists a path P from x to y in G such that each component of $G - V(P)$ has at most three attachments.

Theorem

Suppose all vertices of G have odd degree. If G is Hamiltonian, then G has at least three Hamiltonian cycles.

Q: Find a 3-regular graph with exactly 3 Hamiltonian cycles.

Theorem

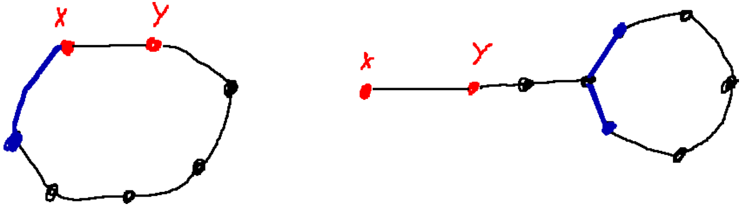
Suppose all vertices of G have odd degree. If G is Hamiltonian, then G has at least three Hamiltonian cycles.

Lemma

If all vertices of G have odd degree, then every edge of G is contained in an even number of Hamiltonian cycles.

For an edge *xy*, an <u>*xy*-lollipop</u> is

- a Hamiltonian cycle containing *xy*, or
- a spanning subgraph consisting of a path starting in *xy* plus a cycle containing the end of the path.

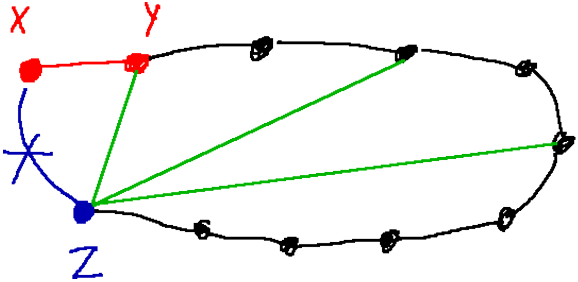


<u>Tails</u> of the *xy*-lollipop:

- the edge of the cycle incident with *x* and different from *xy*
- the edes of the cycle incident with the degree 3 vertex

A graph L:

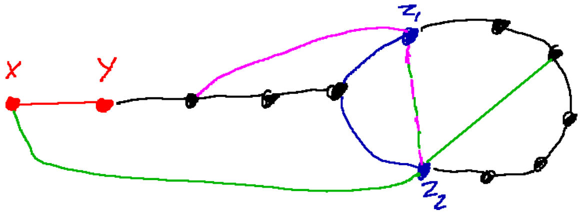
- vertices $=$ xy-lollipops
- H_1 and H_2 adjacent iff H_1 $-$ a tail $=$ H_2 $-$ a tail.



$\deg_L H_1 = \deg_G z - 2$ is odd

A graph L:

- vertices $=$ xy-lollipops
- H_1 and H_2 adjacent iff $H_1 -$ a tail $= H_2 -$ a tail.



$\deg_L H_1 = (\deg_G z_1 - 2) + (\deg_G z_2 - 2)$ is even