Observation

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

Lemma

There exist triangle-free graphs ($\omega = 2$) with arbitrarily large chromatic number.

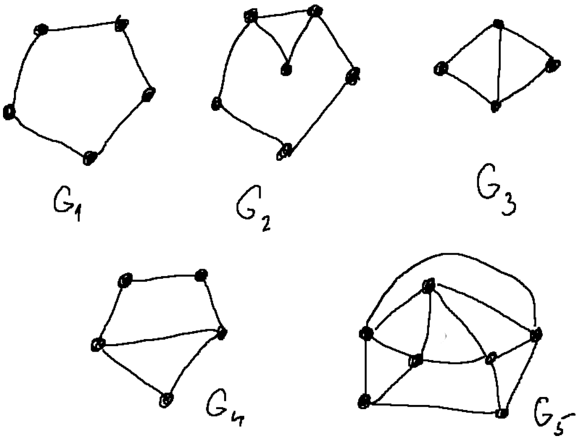
Which graphs satisfy $\chi(G) = \omega(G)$?

- Problem: $\chi(G + K_{|V(G)|}) = \omega(G + K_{|V(G)|}) = |V(G)|$

Definition

A graph G is perfect if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.

Q: Which of the following graphs are perfect?



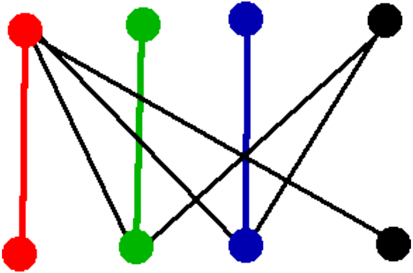
Examples: <u>chordal graphs</u>, bipartite graphs, complements of bipartite graphs, linegraphs of bipartite graphs, comparability graphs, ...

Examples: chordal graphs, <u>bipartite graphs</u>, complements of bipartite graphs, linegraphs of bipartite graphs, comparability graphs, ...

$\chi(G) = \omega(G) = 2$

Examples: chordal graphs, bipartite graphs, complements of bipartite graphs, linegraphs of bipartite graphs, comparability graphs, . . .

$$\chi(\overline{G}) = |V(G)| - \beta(G) = \alpha(G) = \omega(\overline{G})$$

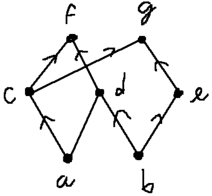


Examples: chordal graphs, bipartite graphs, complements of bipartite graphs, linegraphs of bipartite graphs, comparability graphs, ...

$$\chi(L(G)) = \chi'(G) = \Delta(G) = \omega(L(G))$$

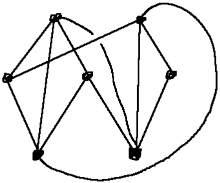
Examples: chordal graphs, bipartite graphs, complements of bipartite graphs, linegraphs of bipartite graphs, comparability graphs, ...

Comparability graph: For a partial ordering $\prec$, $uv \in E(G)$ iff $u \prec v$ or $v \prec u$.



$a < c < f$ $\quad b < d$ $\quad c < g$

$a < d < f$ $\quad b < e < g$



G

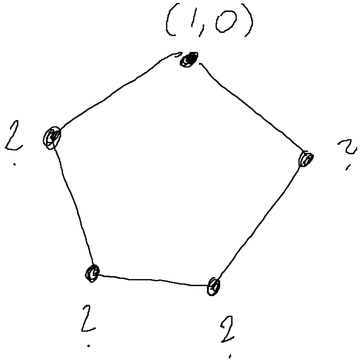
$\omega(G) =$ longest chain, $\chi(G) =$ partition to min. # of antichains.

Definition

For $r \geq 2$, a vector r-coloring is a function $\varphi : V(G) \to$ unit vectors such that for every $uv \in E(G)$,

$$\langle \varphi(u), \varphi(v) \rangle \leq -1/(r-1).$$

Vector $\sqrt{5}$-coloring of C_5; $-1/(\sqrt{5}-1) = \cos\frac{4\pi}{5}$:

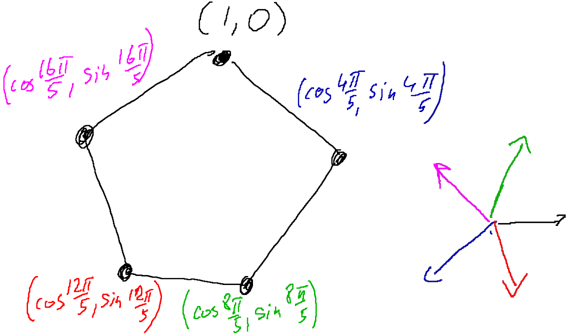


Definition

For $r \geq 2$, a vector r-coloring is a function
$\varphi : V(G) \to$ unit vectors such that for every $uv \in E(G)$,

$$\langle \varphi(u), \varphi(v) \rangle \leq -1/(r-1).$$

Vector $\sqrt{5}$-coloring of C_5; $-1/(\sqrt{5}-1) = \cos \frac{4\pi}{5}$:



Definition

For $r \geq 2$, a vector r-coloring is a function $\varphi : V(G) \to$ unit vectors such that for every $uv \in E(G)$,

$$\langle \varphi(u), \varphi(v) \rangle \leq -1/(r-1).$$

Definition

Vector chromatic number

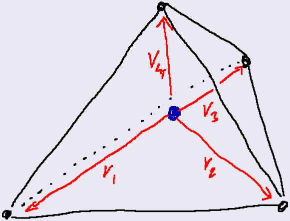
$$\chi_v(G) = \inf\{r \geq 2 : G \text{ has a vector } r\text{-coloring}\}.$$

Lemma

For G with $E(G) \neq \emptyset$, $\chi_v(G) \leq \chi(G)$.

Proof.

$v_1, \ldots, v_c$: unit vectors forming a regular simplex, $\langle v_i, v_j \rangle = s$.



$$0 = \Big| \sum_{i=1}^{c} v_i \Big|^2 = \sum_{i=1}^{c} |v_i|^2 + \sum_{i \neq j} \langle v_i, v_j \rangle = c + c(c-1)s$$

$$s = -1/(c-1)$$

Vertices of color $i \to v_i$. □

Lemma

$\chi_v(G) \geq \omega(G)$

Proof.

$v_1, \ldots, v_k$: Vectors on a clique in a vector r-coloring.

$$
\begin{aligned}
0 &\leq \left| \sum_{i=1}^{k} v_i \right|^2 = \sum_{i=1}^{k} |v_i|^2 + \sum_{i \neq j} \langle v_i, v_j \rangle \\
&\leq k - k(k-1)/(r-1) \\
k &\leq r
\end{aligned}
$$

$\square$

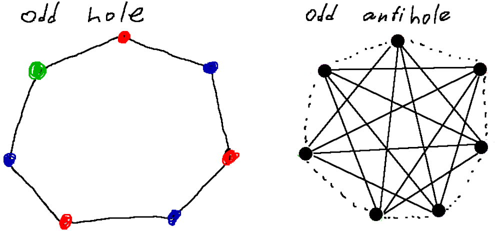
- $\omega(G) \leq \chi_v(G) \leq \chi(G)$
- $\omega(G) = \chi(G) \Rightarrow$ equal to $\chi_v(G)$.
- $\chi_v(G)$ can be computed by semidefinite programming.

$$\chi_v(G) = 1 - 1/t,$$

where t is minimum s.t. there exist vectors $\{v_z : z \in V(G)\}$ satisfying

$$\begin{aligned} \langle v_z, v_z \rangle &= 1 && \text{for every } z \in V(G) \\ \langle v_y, v_z \rangle &\leq t && \text{for every } yz \in E(G) \end{aligned}$$

- Hole: induced ($\geq$4)-cycle
- Antihole: induced subgraph isomorphic to the complement of a ($\geq$4)-cycle



$\chi(C_{2k+1}) = 3$ $\qquad$ $\chi(\overline{C_{2k+1}}) = ?$

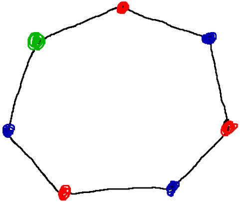
$\omega(C_{2k+1}) = 2$ $\qquad$ $\omega(\overline{C_{2k+1}}) = ?$

Theorem (Strong Perfect Graph Theorem)

A graph is perfect if and only if it contains no odd holes and antiholes.

- **Hole**: induced (≥ 4)-cycle
- **Antihole**: induced subgraph isomorphic to the complement of a (≥ 4)-cycle

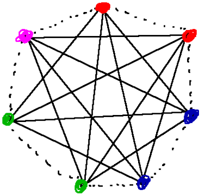
odd hole



$\chi(C_{2k+1}) = 3$

$\omega(C_{2k+1}) = 2$

odd antihole



$\chi(\overline{C_{2k+1}}) = k+1$

$\omega(\overline{C_{2k+1}}) = k$

Theorem (Strong Perfect Graph Theorem)

A graph is perfect if and only if it contains no odd holes and antiholes.

Corollary (Weak Perfect Graph Theorem)

A graph is perfect if and only if its complement is perfect.

Lemma

$A_1, \ldots, A_k, K_1, \ldots, K_k \subseteq \{1, \ldots, n\}$. If

- *$A_i \cap K_i = \emptyset$ for $i \in \{1, \ldots, k\}$ and*
- *$|A_i \cap K_j| = 1$ for $i, j \in \{1, \ldots, k\}$, $i \neq j$,*

then $k \leq n$.

Proof.

- WLOG $k \geq 2$.
- S $k \times n$ matrix, $S_{i,j} = 1$ if $j \in A_i$, 0 otherwise.
- T $n \times k$ matrix, $T_{i,j} = 1$ if $i \in K_j$, 0 otherwise.
- $(ST)_{i,j} = |A_i \cap K_j|$: 0 on the diagonal, 1 elsewhere.
- $k = \text{rank}(ST) \leq \text{rank(S)} \leq n$.

$\square$

Lemma

G is perfect if and only if for every induced subgraph H of G, there exists an independent set intersecting all cliques of size $\omega(H)$.

Proof.

$\Rightarrow$ Clique of size $\omega(H)$ intersects all color classes of $\omega(H)$-coloring.

$\Leftarrow$
- Induction on $|V(G)|$.
- A intersects all cliques of size $\omega(G)$: $\omega(G - A) = \omega(G) - 1$.
- Induction hypothesis: $\chi(G - A) = \omega(G - A) = \omega(G) - 1$.
- Color A using a new color: $\chi(G) = \omega(G)$.

$\square$

Lemma

G is perfect if and only if for every induced subgraph H of G,

$$\alpha(H)\omega(H) \geq |V(H)|.$$

G perfect: $\omega(H) = \chi(H) \geq |V(H)|/\alpha(H)$.

Lemma

G is perfect if and only if for every induced subgraph H of G,

$$\alpha(H)\omega(H) \geq |V(H)|.$$

- For every independent set A, there exists a disjoint $\omega(G)$-clique $K(A)$
- $A_0 = \{v_1, \ldots, v_\alpha\}$ an independent set.
- Induction: $\chi(G - v_i) \leq \omega$; color classes $A_{i,1}, \ldots, A_{i,\omega}$.
  - $K(A_0)$ intersects all of them.
  - $K(A_{i,j})$ intersects all but $A_{i,j}$.
  - $K(A_{i',j})$ for $i' \neq i$ intersects all of them.
    - $K(A_{i',j}) \cap A_0 = \{v_{i'}\} \Rightarrow K(A_{i',j}) \subseteq V(G - v_i)$
- Lemma for sets $A_\star$ and $K(A_\star)$: $\alpha\omega + 1 \leq |V(G)|$ ↯

Lemma

G is perfect if and only if for every induced subgraph H of G,

$$\alpha(H)\omega(H) \geq |V(H)|.$$

If $\alpha(H)\omega(H) \geq |V(H)|$, then

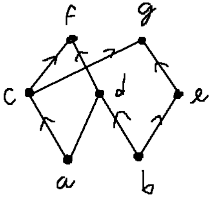
$$\alpha(\overline{H})\omega(\overline{H}) = \omega(H)\alpha(H) \geq |V(H)|.$$

Corollary (Weak Perfect Graph Theorem)

If G is perfect, then $\overline{G}$ is perfect.

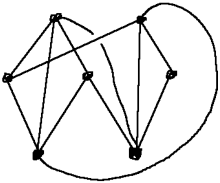
G: the comparability graph of $\prec$

- $\omega(\overline{G}) = \alpha(G) =$ largest antichain in $\prec$
  - The width of $\prec$.
- $\chi(\overline{G}) =$ partition of elements to smallest number of chains



$a < c < f \quad b < d \quad c < g$

$a < d < f \quad b < e < g$



G

Corollary (Dilworth's theorem)

Every finite partially ordered set of width k has a partition into k chains.