Theorem (Vizing)

For any simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

Corollary

For any simple graph G,

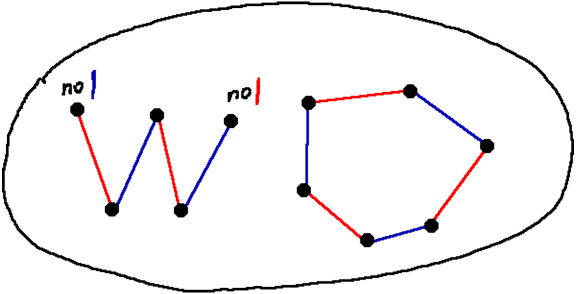
$$\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}.$$

A color c is missing at v if no edge incident with v has color c.

Observation

In an edge coloring by $\Delta(G) + 1$ colors, at least one color is missing at each vertex.

A Kempe chain in colors $\{a, b\}$ is a maximal connected subgraph with edges colored by a or b.

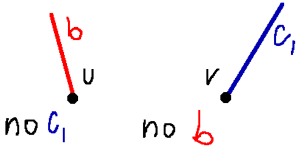


- Alternating path or cycle.
- Path: one of $\{a, b\}$ is missing at each end.
- Switching the chain: Exchanging colors a and b on its edges.
    - Missing colors stay the same, except for the ends of the chain.

Lemma

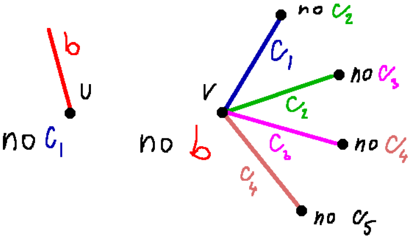
$\chi'(G) \leq \Delta(G) + 1$, *$uv \notin E(G) \Rightarrow$ there exists an edge coloring by $\Delta(G) + 1$ colors s.t. the same color is missing at u and v.*

- c_1: A color missing at u.
- b: A color missing at v.
- WLOG c_1 is not missing at v, b is not missing at u.



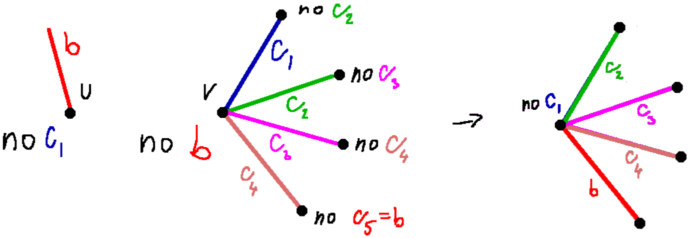
For $i = 1, 2, \ldots$:

- $e_i = vx_i$ an edge of color c_i, c_{i+1} = a color missing at x_i
- If c_{i+1} is missing at v or $c_{i+1} \in \{c_1, \ldots, c_{i-1}\}$:
    - stop and let $k = i$.



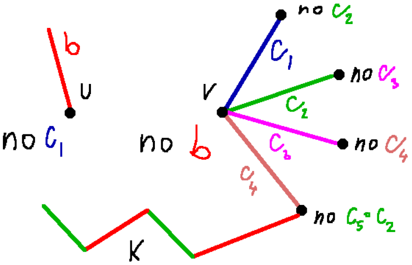
($\star$) If c_{k+1} is missing at v:

- For $i = k, k-1, \ldots, 1$, recolor e_i to c_{i+1}.
- c_1 is missing at both u and v.



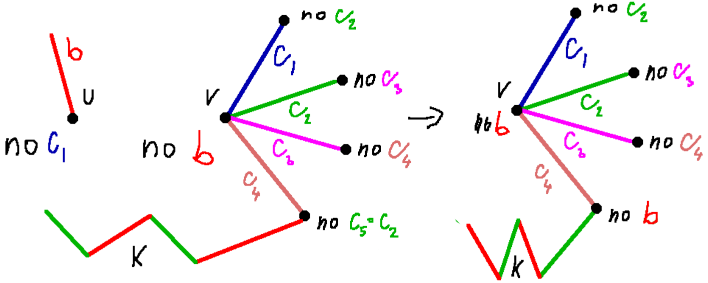
Otherwise: $c_{k+1} = c_s$ for some $s \in \{1, \ldots, k-1\}$.

K: Kempe chain in colors $\{c_s, b\}$ containing x_k



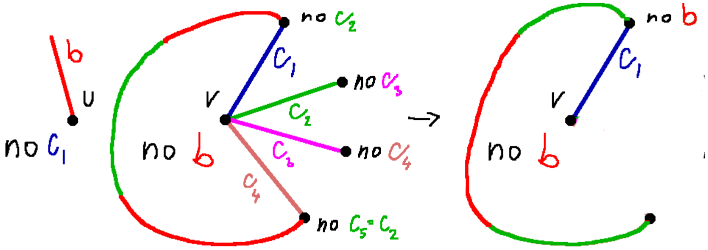
Case 1: K ends at $z \notin \{u, v, x_{s-1}\}$

- Switch K to make b missing at x_k.
- c_{i+1} still missing at x_i for $i = 1, \ldots, k-1$.
- The case $(\star)$ with $c_{k+1} = b$.



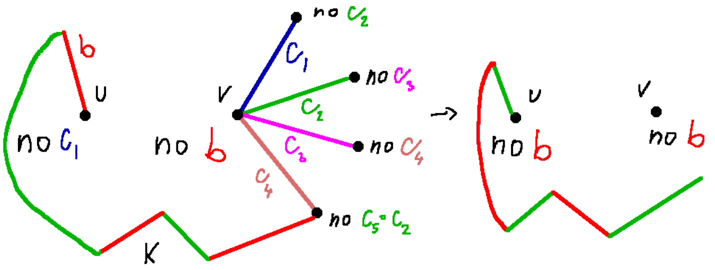
Case 2: K ends at x_{s-1}

- Switch K to make b missing at x_{s-1}.
- The case $(\star)$ with $k = s - 1$, $c_{k+1} = b$.



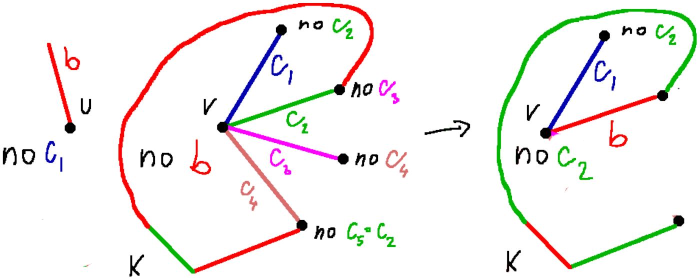
Case 3: *K* ends at *u*

- Switch *K* to make *b* missing at *u*.
- *b* is missing at both *u* and *v*.



Case 4: K ends at v

- K ends by $e_s = vx_s$.
- Switch K to make c_s missing at v.
- The case $(\star)$ with $k = s - 1$, $c_{k+1} = c_s$



Theorem (Vizing)

For any simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

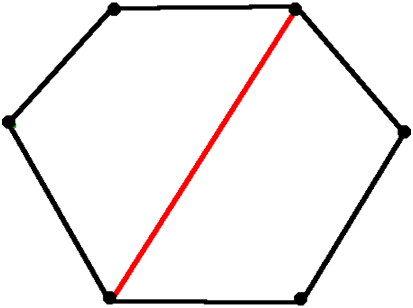
By induction on $|E(G)|$:

- $\chi'(G - uv) \leq \Delta(G - uv) + 1 \leq \Delta(G) + 1$.
- An edge coloring by $\Delta(G) + 1$ colors s.t. c is missing at u and v.
- Color uv by c.

Definition

A graph is chordal if it does not contain any induced cycle of length at least four.

Equivalently, every ($\geq$4)-cycle has a chord.

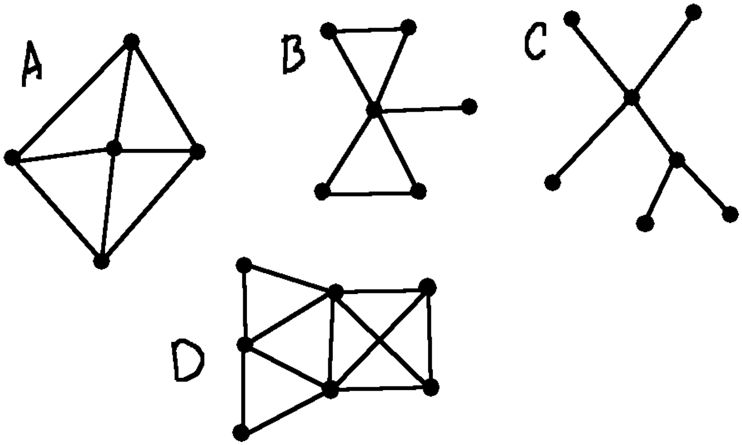


Hole $=$ induced ($\geq$4)-cycle; graph is chordal iff it has no hole.

Definition

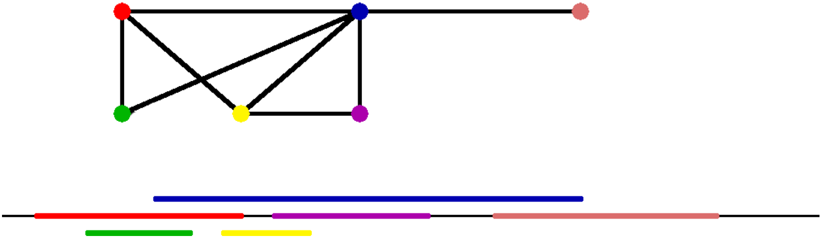
A graph is chordal if it does not contain any induced cycle of length at least four.

Q: Which of the following graphs are chordal?



Example: Interval graphs are chordal.

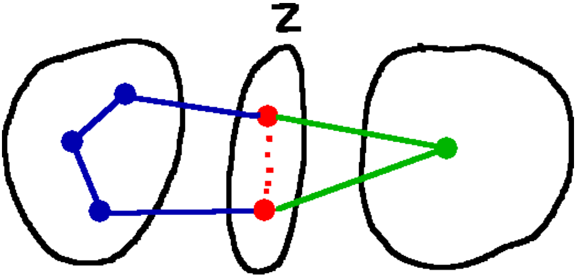
- V = a set of intervals
- $I_1, I_2 \in V$ adjacent iff $I_1 \cap I_2 \neq \emptyset$.



Minimal cut: $G - Z$ not connected, $G - X$ connected for every $X \subsetneq Z$

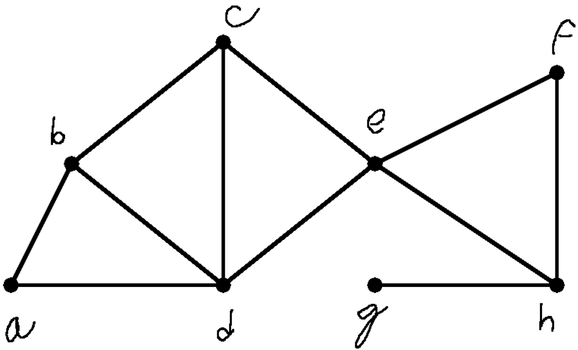
Lemma

If G is chordal, then every minimal cut is a clique.



A vertex is simplicial if its neighborhood is a clique.

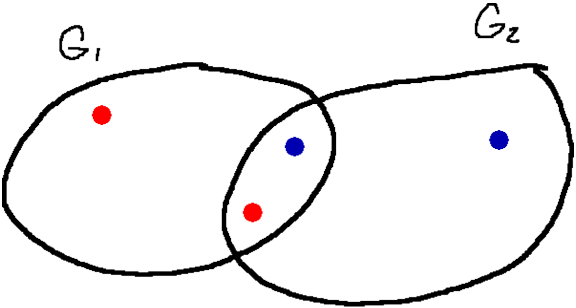
Q: Find simplicial vertices.



Lemma

G chordal, not a clique ⇒ contains two non-adjacent simplicial vertices.

- G not a clique ⇒ contains a minimal cut.
- Induction for the sides of the cut.



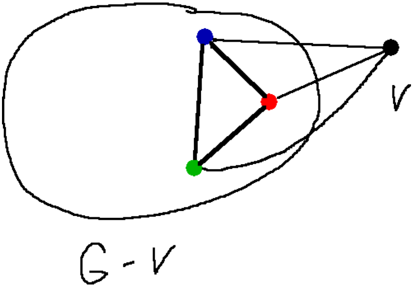
Corollary

A graph G is chordal if and only if every induced subgraph of G contains a simplicial vertex.

- Induced subgraphs of chordal graphs are chordal.
- (≥ 4)-cycle does not have a simplicial vertex.

If $v \in V(G)$ is simplicial, then

- $\chi(G) = \max(\chi(G - v), \deg v + 1)$
- $\omega(G) = \max(\omega(G - v), \deg v + 1)$
- $\alpha(G) = \alpha(G - N[V]) + 1$



Corollary

If G is chordal, then

- $\chi(G) = \omega(G)$
- $\chi(G)$, $\omega(G)$ *and* $\alpha(G)$ *can be computed in polynomial time.*

An elimination ordering is an ordering $v_1, \ldots, v_n$ of vertices of G such that for $i = 1, \ldots, n$,

$$\{v_j : j < i, v_j v_i \in E(G)\} \text{ is a clique.}$$

Q: Show that every chordal graph has an elimination ordering.

Lemma

If G has an elimination ordering, then G is chordal.

- Every induced subgraph of G has an elimination ordering.
- The last vertex of an elimination ordering is simplicial.

Corollary

To test whether G is chordal, delete simplicial vertices in any order, until we obtain either

- *an elimination ordering of G, or*
- *an induced subgraph with no simplicial vertex.*

Corollary

A graph is chordal iff it is obtained from a single-vertex graph by repeatedly adding simplicial vertices.