Definition

For a surface Σ,

$$\chi(\Sigma) = \max\{\chi(G) : G \text{ can be drawn in } \Sigma\}$$

$$\omega(\Sigma) = \max\{\omega(G) : G \text{ can be drawn in } \Sigma\}$$

Q: Determine χ(sphere) and ω(sphere).

Observation

$\chi(\Sigma) \geq \omega(\Sigma)$

Observation

If G is drawn in Σ, then $K_{\omega(\Sigma)+1} \not\preceq_m G$.

If Hadwiger's conjecture is true, then this implies

- $\chi(G) \leq \omega(\Sigma)$
- $\chi(\Sigma) = \omega(\Sigma)$

Goal: Prove $\chi(\Sigma) = \omega(\Sigma)$ without Hadwiger's conjecture.

Lemma (A)

If G is an n-vertex graph ($n \geq 3$) drawn in a surface of Euler genus g, then $\delta(G) \leq \min(n-1, 6+6(g-2)/n)$.

Proof.

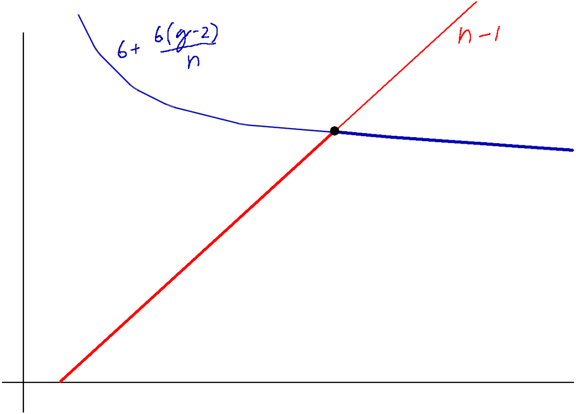
- Last time, we proved $|E(G)| \leq 3n + 3g - 6$.
- G has average degree $\frac{2|E(G)|}{n} \leq 6 + 6(g-2)/n$.

$\square$

Lemma (B)

If $g \geq 2$, then for every n,

$$\min(n-1, 6+6(g-2)/n) \leq \frac{5+\sqrt{24g+1}}{2}.$$



Lemma (B)

If $g \geq 2$, then for every n,

$$\min(n-1, 6+6(g-2)/n) \leq \frac{5+\sqrt{24g+1}}{2}.$$

Proof.

The expression is maximized when

$$\begin{aligned} n-1 &= 6+6(g-2)/n \\ n^2-7n-6(g-2) &= 0 \\ n &= \frac{7+\sqrt{24g+1}}{2}. \end{aligned}$$

$\square$

Let

$$H(g) = \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor.$$

Lemma

If G is drawn in a surface of Euler genus $g > 0$, then $\delta(G) < H(g)$.

Proof.

- $g \geq 2$:

$$\delta(G) \leq \frac{5 + \sqrt{24g + 1}}{2} < H(g)$$

  by Lemmas (A) and (B).
- $g = 1$: $\delta(G) < 6 = H(1)$ by Lemma (A).

$\square$

Theorem

If G is drawn in a surface of Euler genus g, then $\chi(G) \leq H(g)$.

Proof.

- $g = 0$: $\chi(G) \leq 4 = H(0)$ by the Four Color Theorem.
- $g > 0$: By induction;
  - $v \in V(G)$ s.t. $\deg v < H(g)$
  - Color $G - v$ by $H(g)$ colors by the induction hypothesis.
  - Extend the coloring to v.

$\square$

Theorem (Ringel and Youngs)

If $\Sigma \neq$ Klein bottle is a surface of Euler genus g, then $K_{H(g)}$ can be drawn in Σ.

Corollary

For every surface $\Sigma \neq$ Klein bottle,

$$\chi(\Sigma) = H(g) = \omega(\Sigma).$$

Lemma

ω(*Klein bottle*) $= 6$.

Observation

Every graph G drawn in the Klein bottle has average degree at most 6. Hence, either

- $\delta(G) \leq 5$, *or*
- *G is* 6*-regular.*

Observation

If G is a graph of maximum degree Δ, then $\chi(G) \leq \Delta + 1$.

Q: Find a graph of maximum degree Δ that cannot be colored by Δ colors.

Theorem (Brooks)

If G is a connected graph of maximum degree Δ and G is neither a clique nor an odd cycle, then $\chi(G) \leq \Delta$.

Corollary

Every graph drawn G in the Klein bottle is 6-colorable.

- $\deg v \leq 5$:
  - Color $G - v$ by induction hypothesis, extend to v.
- G 6-regular:
  - $G \neq K_7$, since it is drawn in the Klein bottle.
  - 6-colorable by Brooks theorem.

Corollary

For a surface Σ of Euler genus g:

- *If $\Sigma \neq$ Klein bottle, then*

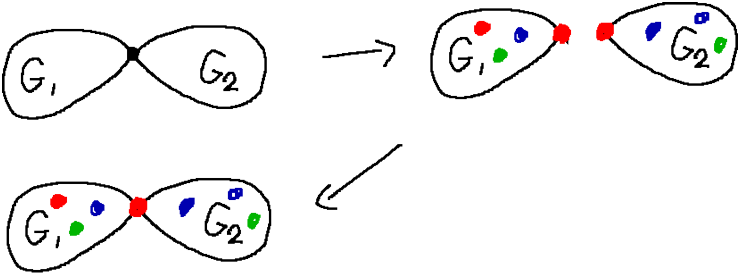
$$\chi(\Sigma) = \omega(\Sigma) = H(g).$$

- *If $\Sigma =$ Klein bottle, then*

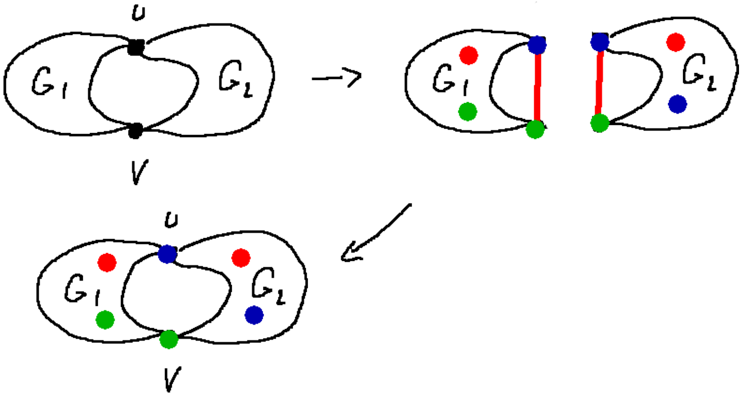
$$\chi(\Sigma) = \omega(\Sigma) = 6 = H(g) - 1.$$

- Proof of Brooks theorem:
  - $\Delta \leq 2$: Simple.
  - $\Delta \geq 3$: By induction on $|V(G)|$.

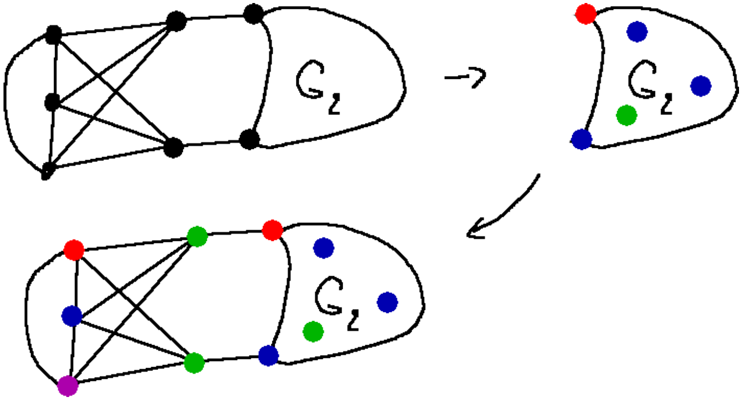
Case 1: G is not 2-connected:



Case 2: G is 2-connected but not 3-connected:
(a) $G_1 + uv, G_2 + uv \neq K_{\Delta+1}$:

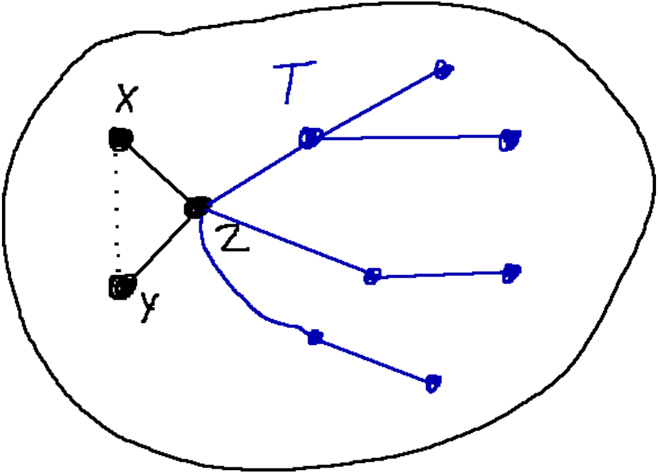


Case 2: G is 2-connected but not 3-connected:
(b) $G_1 + uv = K_{\Delta+1}$:



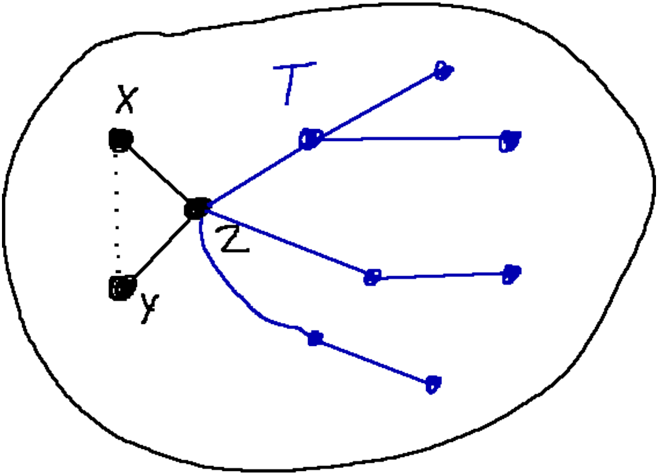
Case 3: G is 3-connected:

- x and y: vertices at distance 2
- z: Common neighbor of x and y
- T: Spanning tree of $G - \{x, y\}$ plus edges xz, yz



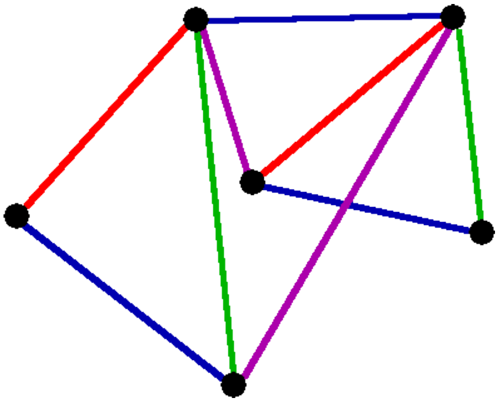
Case 3: G is 3-connected:

- Root T in z.
- Give x and y color 1.
- Color in T from leaves up.



Definition

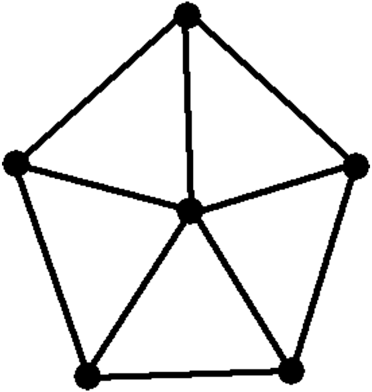
$\varphi : E(G) \to \{1, \ldots, k\}$ is an edge k-coloring if $\varphi(e_1) \neq \varphi(e_2)$ for distinct $e_1, e_2 \in E(G)$ incident with the same vertex.



Definition

The chromatic index $\chi'(G)$: the minimum k such that G has an edge k-coloring.

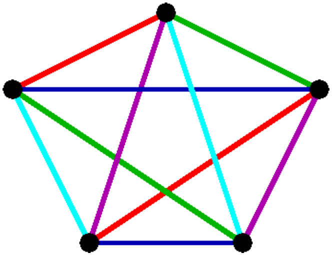
Q: What is the chromatic index of the following graph?



Example:

- Tournament with n players.
- Each two need to play a match.
- Any number of matches can be played in parallel.
- A player can only play one match in a round.

$$\text{min. \# of rounds} = \chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$



Observation

$$\chi'(G) \geq \Delta(G)$$

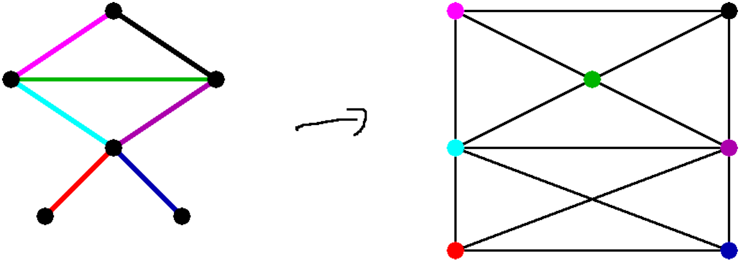
Observation

$$\chi'(G) \geq \frac{|E(G)|}{\textit{size of maximum matching in G}}$$

Definition

The linegraph $L(G)$ of G has

- $V(L(G)) = E(G)$
- $e_1 e_2 \in E(L(G))$ iff e_1 and e_2 are incident with the same vertex.



Observation

$\chi'(G) = \chi(L(G))$

Not every graph is a linegraph!

Claim

There is no G such that $L(G) = K_{1,3}$.

Observation

$$\Delta(L(G)) \leq 2\Delta(G) - 2$$

Corollary

- $\chi'(G) \leq 2\Delta(G) - 1$
- *If G is connected and $L(G)$ is neither a clique nor an odd cycle, then $\chi'(G) \leq 2\Delta(G) - 2$.*

Theorem (Vizing)

For any simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

Corollary

For any simple graph G,

$$\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}.$$