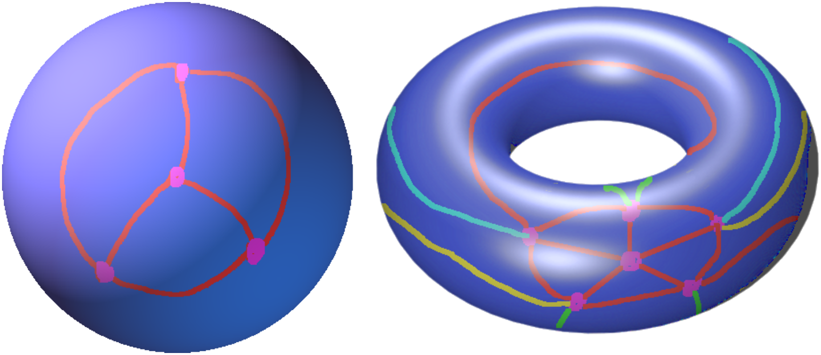


- Locally indistinguishable from the plane.
  - Every point has a neighborhood that looks like an open disk.
- Not too complicated: can be covered by finitely many such small neighborhoods.
- Connected.

Definition

A surface is a compact connected 2-dimensional manifold without a boundary.

- Locally indistinguishable from the plane.
  - Every point has a neighborhood that looks like an open disk.
- Not too complicated: can be covered by finitely many such small neighborhoods.
- Connected.

Q: Which of the following spaces are surfaces?

- sphere
- torus
- plane
- closed disk (circle and its interior)

Definition

A function $f : X \to Y$ between topological spaces is a homeomorphism if f is bijective and both f and f^{-1} are continuous. If there exists a homeomorphism from X to Y, we say X and Y are homeomorphic.

Observation

If f is a homeomorphism of surfaces, then

- $f(\textit{simple continuous curve}) = \textit{simple continuous curve}$
- $f(\textit{drawing of } G) = \textit{drawing of } G$

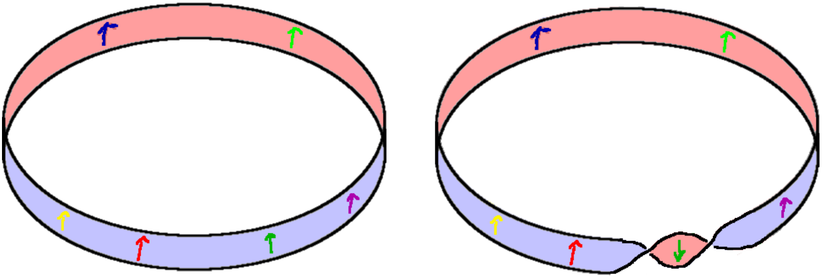
Definition

A function $f : X \to Y$ between topological spaces is a homeomorphism if f is bijective and both f and f^{-1} are continuous. If there exists a homeomorphism from X to Y, we say X and Y are homeomorphic.

- Torus and the surface of a coffee mug are homeomorphic.
- The cylinder and the twice-twisted band are homeomorphic.
- The cylinder and the once-twisted band (the Möbius band) are not homeomorphic.

Definition

A function $f : X \to Y$ between topological spaces is a homeomorphism if f is bijective and both f and f^{-1} are continuous. If there exists a homeomorphism from X to Y, we say X and Y are homeomorphic.



Definition

A function $f : X \to Y$ between topological spaces is a homeomorphism if f is bijective and both f and f^{-1} are continuous. If there exists a homeomorphism from X to Y, we say X and Y are homeomorphic.

- Torus and the surface of a coffee mug are homeomorphic.
- The cylinder and the twice-twisted band are homeomorphic.
- The cylinder and the once-twisted band (the Möbius band) are not homeomorphic.

Definition

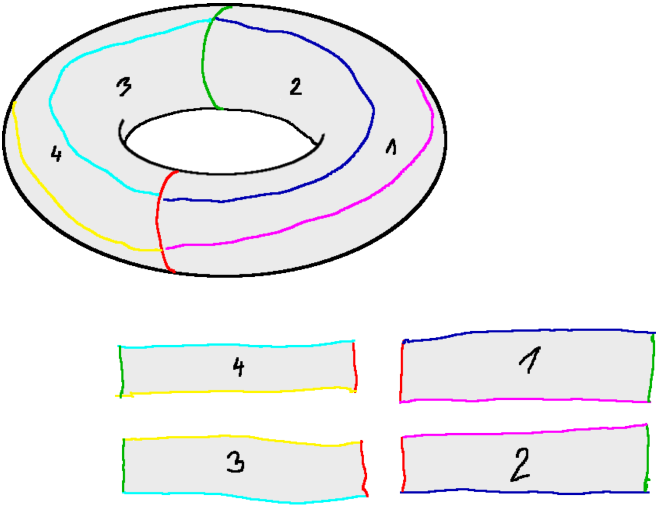
A net of a surface is a graph drawn in the surface so that every face is homeomorphic to an open disk.

Claim

Every surface has a net.

We can

- cut the surface along a net and
- glue it back together from the resulting polygons.

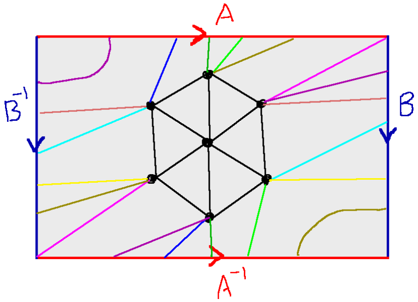


Observation

If G is a net and e is incident with two different faces of G, then $G - e$ is a net.

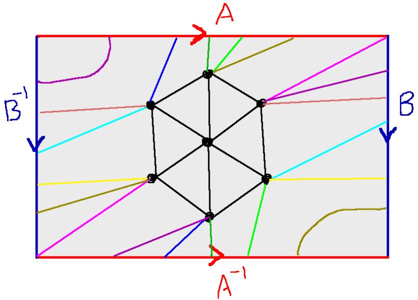
Corollary

Every surface has a net with only one face. Equivalently, every surface can be obtained by gluing pairs of edges on a single polygon.



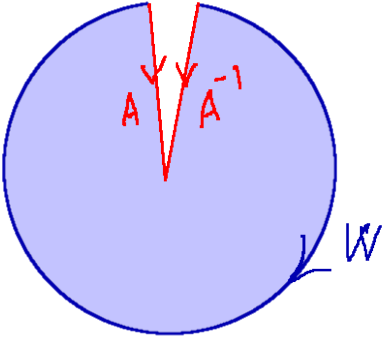
- Arrows indicate the direction of gluing.
- A: Clockwise arrow, A^{-1}: counterclockwise arrow.

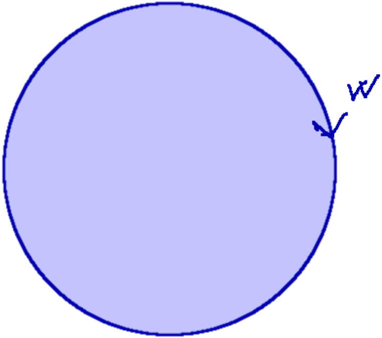
$ABA^{-1}B^{-1}$:



Observation

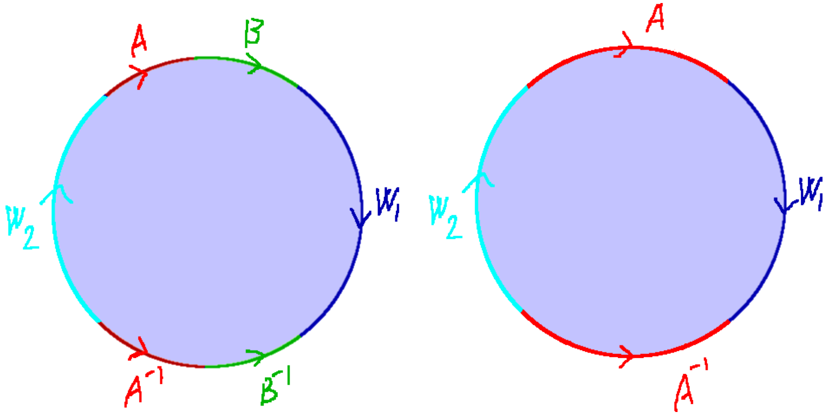
- $AA^{-1}w$ and w represent the same surface.
- $ABw_1B^{-1}A^{-1}w_2$ and $Aw_1A^{-1}w_2$ represent the same surface.
- Aw_1Aw_2 and $AAw_2w_1^{-1}$ represent the same surface.





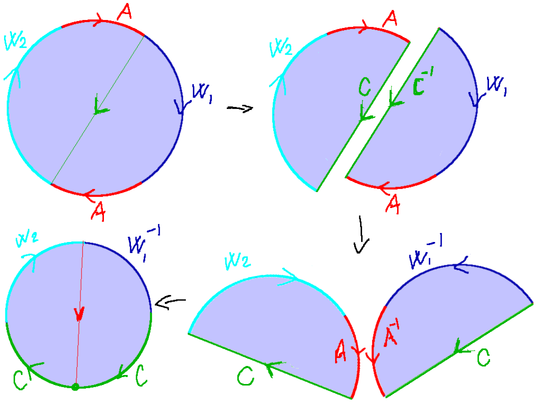
Observation

- $AA^{-1}w$ and w represent the same surface.
- $ABw_1B^{-1}A^{-1}w_2$ and $Aw_1A^{-1}w_2$ represent the same surface.
- Aw_1Aw_2 and $AAw_2w_1^{-1}$ represent the same surface.



Observation

- $AA^{-1}w$ and w represent the same surface.
- $ABw_1B^{-1}A^{-1}w_2$ and $Aw_1A^{-1}w_2$ represent the same surface.
- Aw_1Aw_2 and $AAw_2w_1^{-1}$ represent the same surface.



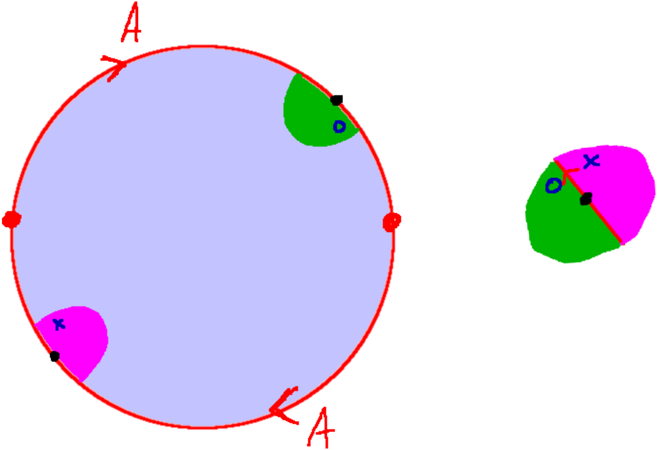
Observation

- *$AA^{-1}w$ and w represent the same surface.*
- *$ABw_1B^{-1}A^{-1}w_2$ and $Aw_1A^{-1}w_2$ represent the same surface.*
- *Aw_1Aw_2 and $AAw_2w_1^{-1}$ represent the same surface.*

Q: Two of these expressions represent the same surface; which two?

$$ABA^{-1}B^{-1}, \; ABA^{-1}B, \; AABB$$

Does *AA* represent a surface?



Lemma

If G_1 and G_2 are nets of the same surface, then

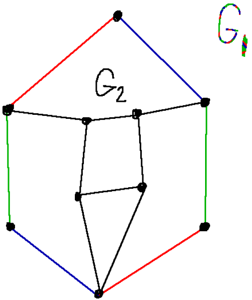
$$|E(G_1)| - |V(G_1)| - |F(G_1)| = |E(G_2)| - |V(G_2)| - |F(G_2)|.$$

WLOG:

- Drawings of G_1 and G_2 intersect in finite number of points (nontrivial!)
- G_1 intersects G_2 only in vertices
  - subdividing edges: both $|E|$ and $|V|$ increase by 1
- $G_1 \subseteq G_2$
  - Compare G_1 vs. $G_1 \cup G_2$ vs. G_2
- G_1 has exactly one face
  - deleting edge between distinct faces: both $|E|$ and $|F|$ decrease by 1

G': Plane graph obtained by taking

- polygonal representation corresponding to the net G_1 and
- the drawing of G_2 in the polygon.



$$
\begin{aligned}
|F(G')| &= |F(G_2)| + 1 = |F(G_2)| - |F(G_1)| + 2 \\
|E(G')| &= |E(G_2)| + |E(G_1)| \\
|V(G')| &= |V(G_2)| - |V(G_1)| + 2|E(G_1)| \\
0 &= |E(G')| - |F(G')| - |V(G')| + 2 \\
&= (|E(G_2)| - |V(G_2)| - |F(G_2)|) \\
&\quad - (|E(G_1)| - |V(G_1)| - |F(G_1)|)
\end{aligned}
$$

Definition

The Euler genus of a surface with a net G is

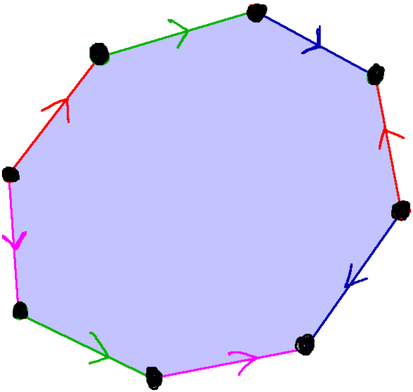
$$|E(G)| - |V(G)| - |F(G)| + 2.$$

Observation

Let G be a net of a surface Σ with exactly one face.

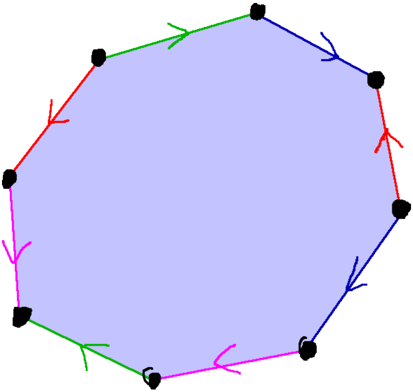
- *If $\Sigma =$ sphere, then G is a tree, and thus the sphere has Euler genus 0.*
- *Otherwise, if $|V(G)|$ is minimum possible, then G has minimum degree at least two, and thus Σ has positive Euler genus.*

Example:



4 edges, 1 face, 2 vertices → Euler genus 3.

Q: Determine the Euler genus of the following surface:



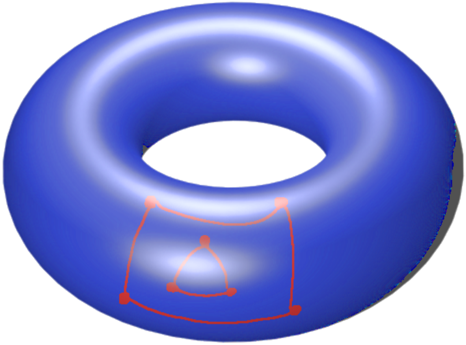
Theorem (Generalized Euler formula)

If a graph G is drawn in a surface of Euler genus g, then

$$|E(G)| \leq |V(G)| + |F(G)| + g - 2.$$

Proof.

Add edges to G to extend it to a net. □



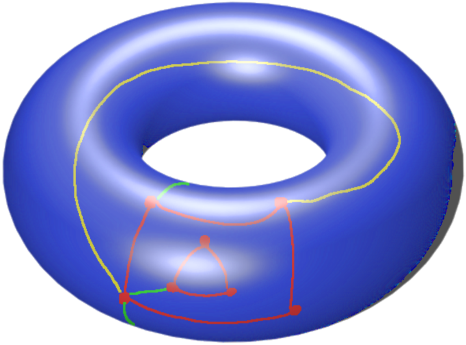
Theorem (Generalized Euler formula)

If a graph G is drawn in a surface of Euler genus g, then

$$|E(G)| \leq |V(G)| + |F(G)| + g - 2.$$

Proof.

Add edges to G to extend it to a net. □



Theorem (Generalized Euler formula)

If a graph G is drawn in a surface of Euler genus g, then

$$|E(G)| \leq |V(G)| + |F(G)| + g - 2.$$

Proof.

Add edges to G to extend it to a net. □

Corollary

If a simple graph G is drawn in a surface of Euler genus g and $|V(G)| \geq 3$*, then* $|E(G)| \leq 3|V(G)| + 3g - 6$*.*

Proof.

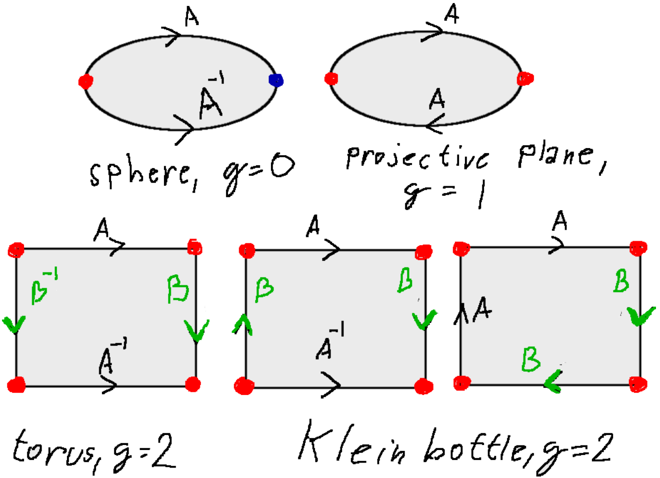
$$2|E(G)| \geq 3|F(G)| \geq 3(|E(G)| - |V(G)| - g + 2)$$

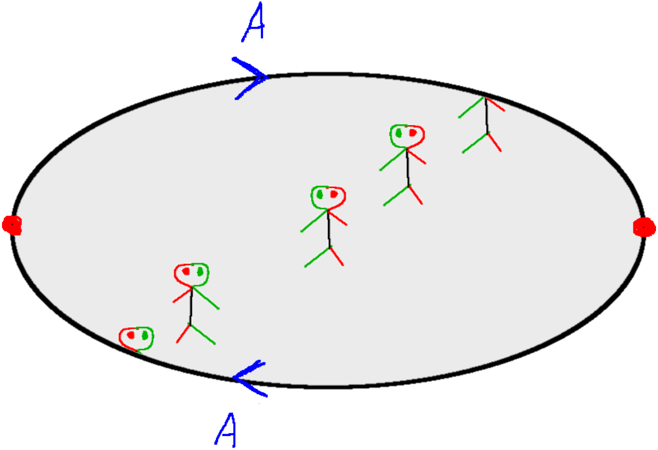
□

Corollary

K_8 cannot be drawn in the torus (Euler genus $g = 2$).

$$|E(G)| = 28 > 24 = 3|V(G)| + 3g - 6$$


A
A⁻¹
sphere, g=0
A
A
projective plane,
g = 1
A
B⁻¹
B
A⁻¹
torus, g=2
A
B
B
A⁻¹
A
B
A
B
Klein bottle, g=2

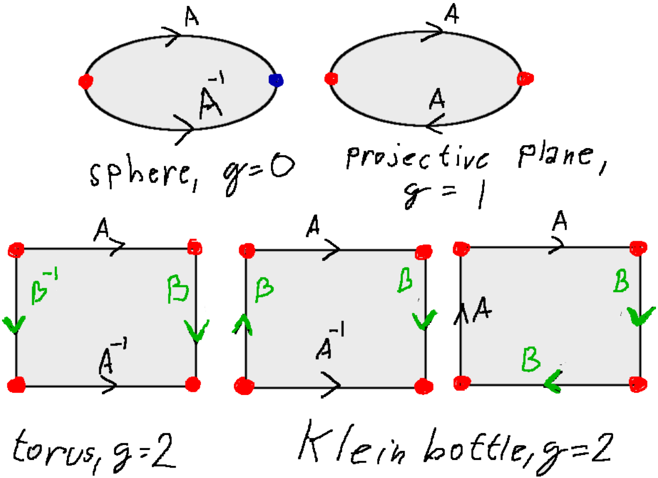

A
A

Definition

A surface is orientable if you can consistently define orientation at every point in the surface, and non-orientable otherwise.

Observation

A surface is non-orientable if and only if it is represented by an expression of form Aw_1Aw_2.


A
A^{-1}
sphere, g=0
A
A
projective plane, g = 1
A
B^{-1}
B
A^{-1}
torus, g=2
A
B
B
A^{-1}
A
B
A
B
Klein bottle, g=2

Theorem (Classification theorem)

Every surface has a representation in one of the two following forms:

- $(ABA^{-1}B^{-1})(CDC^{-1}D^{-1})\dots$
  - *orientable, k blocks $\rightarrow$ Euler genus $2k$*
- $(AA)(BB)(CC)\dots$
  - *non-orientable, k blocks $\rightarrow$ Euler genus k*

Corollary

Two surfaces are homeomorphic if and only if they have the same Euler genus and orientability.

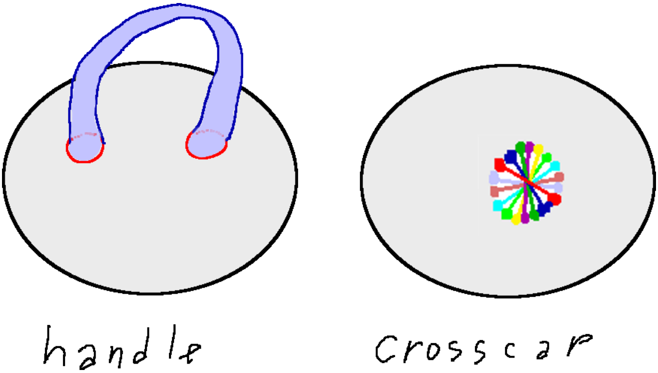
Corollary

The Euler genus of an orientable surface is always even.

The genus of a surface Σ:

- Euler genus$/2$ if Σ is orientable.
- Euler genus if Σ is non-orientable.

Another approach: Start with a sphere, then add handles and crosscaps:



Observation

A surface obtained from the sphere by adding a handles and b crosscaps has Euler genus $2a + b$, and it is orientable iff $b = 0$. Consequently, every surface is homeomorphic to some surface obtained in this way.