Q: State the Kuratowski's theorem about planar graphs.

Q: State the Kuratowski's theorem about planar graphs.

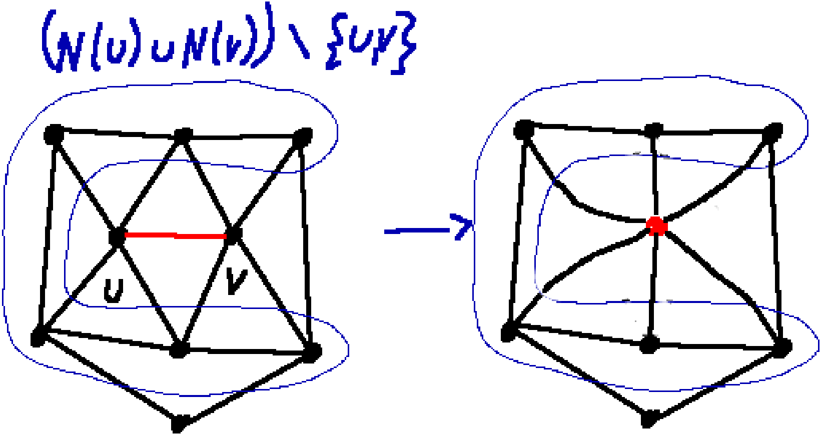
G contains a subdivision of a graph H as a subgraph:

- H is a topological minor of G
- $H \preceq_t G$

Theorem (Kuratowski)

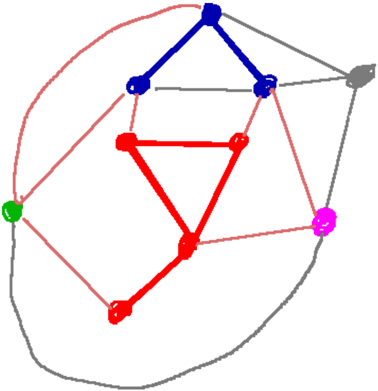
A graph G is planar if and only if $K_5, K_{3,3} \not\preceq_t G$.

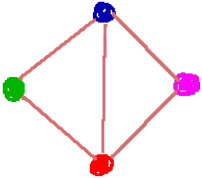
Contraction of an edge uv in a graph G, gives the graph G/uv:



Definition

H is a minor of *G* if *H* is obtained from *G* by edge contractions and edge and vertex deletions. We write $H \preceq_m G$.

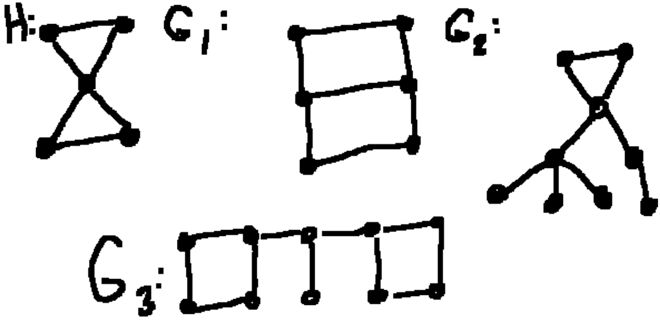




Definition

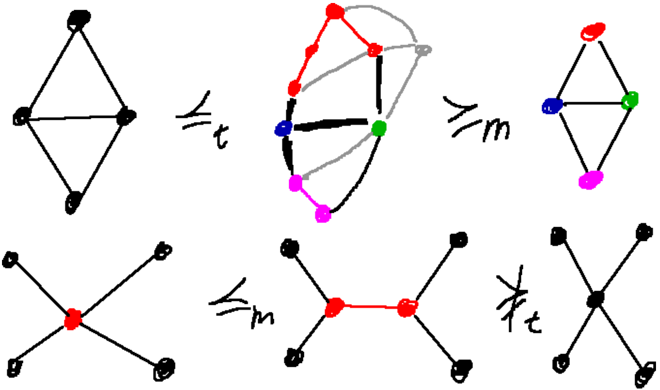
H is a minor of *G* if *H* is obtained from *G* by edge contractions and edge and vertex deletions. We write $H \preceq_m G$.

Q: Which of the following graphs contain *H* as a minor?



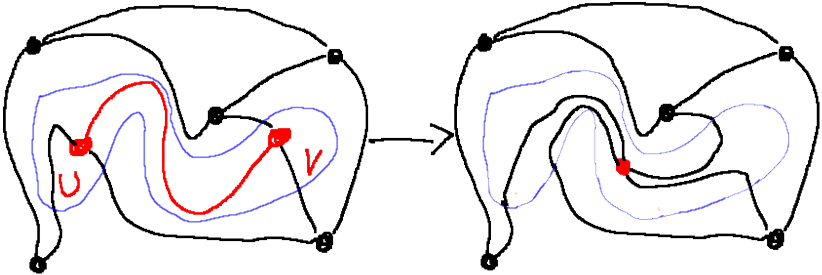
Observation

- *If $A \preceq_t G$, then $A \preceq_m G$.*
- *If $A \subseteq G$, then $A \preceq_m G$.*
- *If $F \preceq_m H$ and $H \preceq_m G$, then $F \preceq_m G$.*
- *$A \preceq_m G$ does not imply $A \preceq_t G$.*



Observation

If G is planar, then all minors of G are planar, and in particular $K_5, K_{3,3} \not\preceq_m G$.



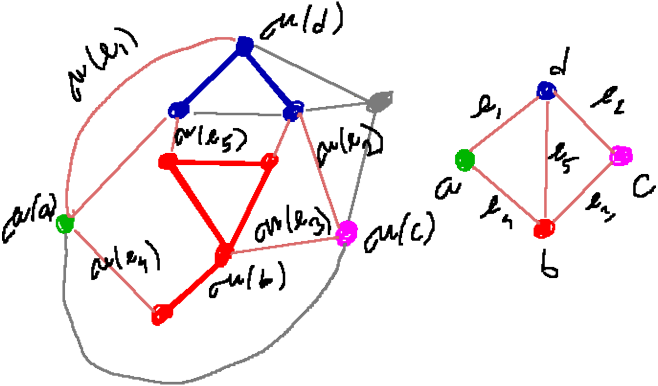
Theorem (Wagner)

A graph G is planar if and only if $K_5, K_{3,3} \not\preceq_m G$.

Definition

μ is a model of H in G if

- $\forall v \in V(H)$: $\mu(v)$ is a connected subgraph of G
- $u \neq v \Rightarrow V(\mu(u)) \cap V(\mu(v)) = \emptyset$
- $\forall uv \in E(H)$: $\mu(uv) = xy \in E(G)$ such that $x \in V(\mu(u))$ and $y \in V(\mu(v))$.

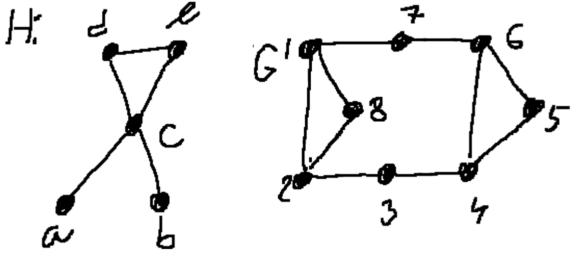


Definition

μ is a model of H in G if

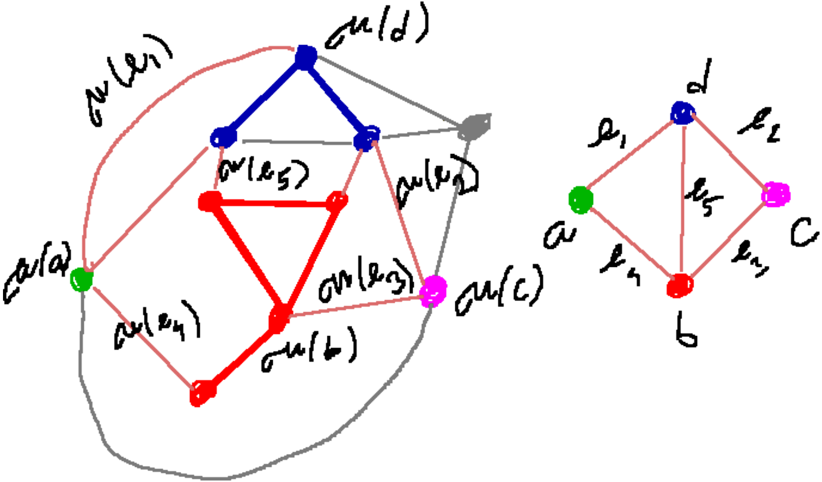
- $\forall v \in V(H)$: $\mu(v)$ is a connected subgraph of G
- $u \neq v \Rightarrow V(\mu(u)) \cap V(\mu(v)) = \emptyset$
- $\forall uv \in E(H)$: $\mu(uv) = xy \in E(G)$ such that $x \in V(\mu(u))$ and $y \in V(\mu(v))$.

Q: Describe a model of H in G:



Lemma

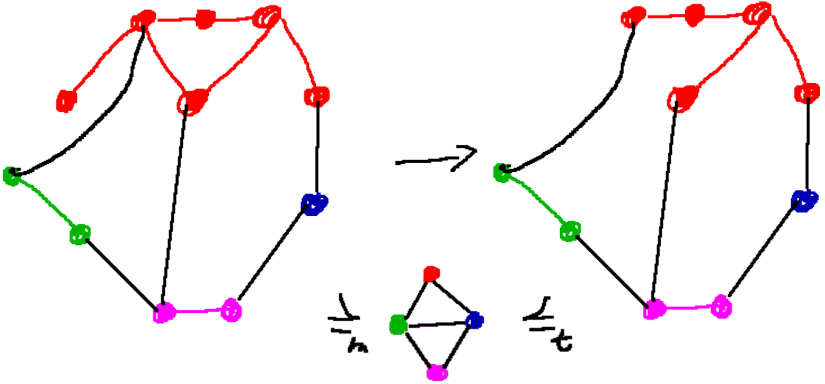
$H \preceq_m G$ iff there exists a model of H in G.



Lemma

When $\Delta(H) \leq 3$*:*

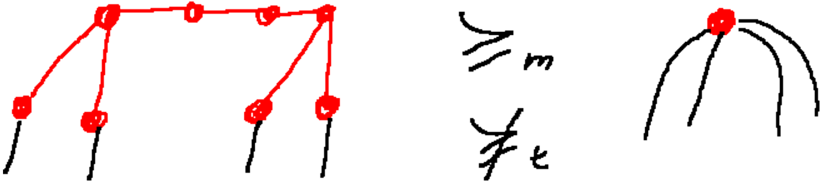
$$H \preceq_m G \text{ iff } H \preceq_t G.$$



Lemma

When $\Delta(H) \leq 3$*:*

$$H \preceq_m G \text{ iff } H \preceq_t G.$$



Lemma

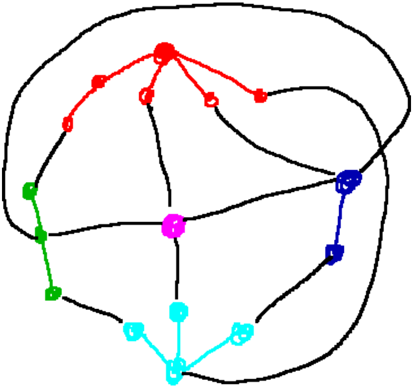
When $\Delta(H) \leq 3$*:*

$$H \preceq_m G \text{ iff } H \preceq_t G.$$

Q: Which graphs do not contain K_3 as a minor?

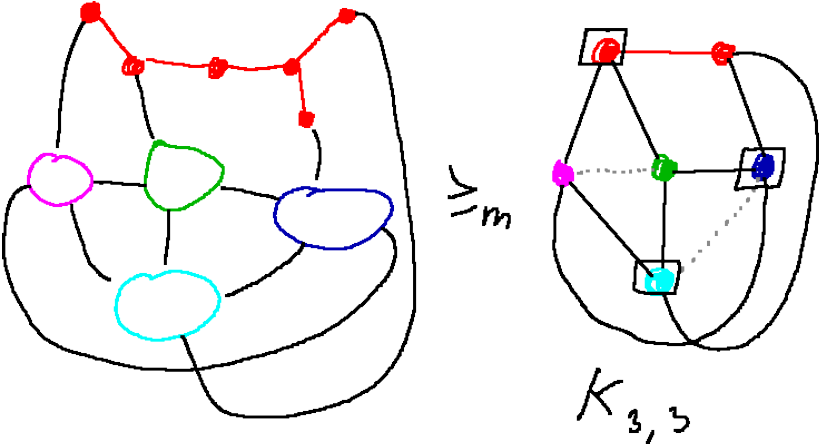
Lemma

If $K_5 \preceq_m G$, then $K_5 \preceq_t G$ or $K_{3,3} \preceq_t G$.



Lemma

If $K_5 \preceq_m G$, then $K_5 \preceq_t G$ or $K_{3,3} \preceq_t G$.



Lemma

Wagner's theorem implies Kuratowski's theorem.

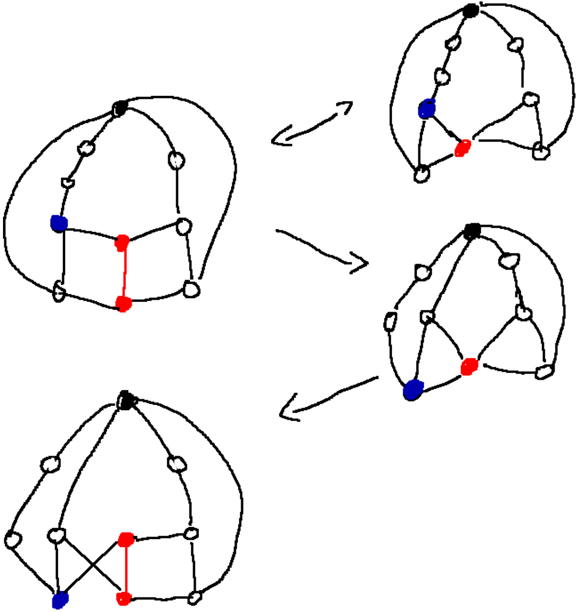
Proof.

- Suppose $K_5, K_{3,3} \not\preceq_t G$.
- Then $K_5, K_{3,3} \not\preceq_m G$.
- G planar by Wagner's theorem.

□

Wagner's theorem, proof idea (by induction on $|V(G)|$):

- Choose $uv \in E(G)$.
- Since $K_5, K_{3,3} \not\preceq_m G$, we have $K_5, K_{3,3} \not\preceq_m G/uv$
- By the induction hypothesis, G/uv is planar.
- Decontract the edge uv in the drawing of G/uv.



Definition

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism if it is bijective, continuous, and f^{-1} is continuous.

Q: Give an example of a homeomorphism of the plane.

Theorem

If G is planar and 3-connected, then any two drawings of G in the plane differ only by a homeomorphism.

- Fix: First prove Wagner's theorem for 3-connected graphs, then deal with (≤ 2)-cuts.
- Problem: We need to ensure G/uv is 3-connected.

Theorem (Tutte)

If $G \neq K_4$ is 3-connected, then there exists $e \in E(G)$ such that G/e is 3-connected.

Corollary

Every 3-connected graph can be obtained from K_4 by decontracting edges.

Compare:

Lemma

Every 2-connected graph can be obtained from a cycle by adding ears.