For even 2-factor F:

- We defined $\mathrm{sgn}(F) = (-1)^{c(F)}$.
- The following are the same:
  - The number of permutations with cycles F.
  - The number of pairs (M_1, M_2) of perfect matchings such that $F = M_1 + M_2$.

$\lambda(x, y) = 1$ if $x < y$ and -1 if $x > y$

- $C = v_1 \ldots v_t$ even cycle: $\lambda(C) = \lambda(v_1, v_2) \cdot \lambda(v_2, v_3) \cdots \lambda(v_t, v_1)$.
- F even 2-factor:

$$\lambda(F) = \prod_{C \text{ cycle of } F} \lambda(C).$$

We defined the sign of a perfect matching so that:

Lemma

$$\mathrm{sgn}(M_1)\mathrm{sgn}(M_2) = \mathrm{sgn}(M_1 + M_2)\lambda(M_1 + M_2)$$

For $b : E(G) \to \mathbb{R}$, the Pfaffian of (G, b) is

$$\mathrm{Pf}(G, b) = \sum_{M \text{ perfect matching of } G} \mathrm{sgn}(M) \prod_{e \in E(M)} b(e).$$

Pfaffian function: $b : E(G) \to \{-1, 1\}$ such that

$$\mathrm{sgn}(M) \cdot \prod_{e \in E(M)} b(e)$$

is the same for every perfect matching M of G.

Observation

If b is a Pfaffian function, then

$$|\mathrm{Pf}(G, b)| = \textit{number of perfect matchings in } G.$$

Lemma

For any graph G and a function $b : E(G) \to \mathbb{Z}$, $|\mathrm{Pf}(G, b)|$ can be computed in polynomial time.

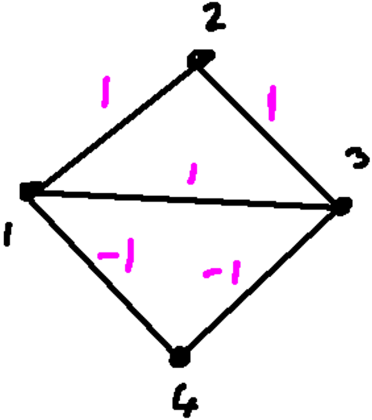
Theorem (Kasteleyn)

For every planar graph G, we can find a Pfaffian function b in polynomial time.

Corollary

Polynomial-time algorithm to find the number of perfect matchings in a planar graph G.

The antisymmetric adjacency matrix of (G, b):
$C_{u,v} = b(uv)\lambda(u, v)$ for $uv \in E(G)$, 0 otherwise.



$$C = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The antisymmetric adjacency matrix of (G, b):
$C_{u,v} = b(uv)\lambda(u, v)$ for $uv \in E(G)$, 0 otherwise.

Terms for non-even-cycled permutations in $\det(C)$ cancel:

- π' obtained from π by reversing an odd cycle K.

$$\begin{aligned}
\operatorname{sgn}(\pi') \prod_{i\in\{1,\ldots,n\}} C_{i,\pi'(i)} &= \operatorname{sgn}(\pi) \prod_{i\in\{1,\ldots,n\}\setminus V(K)} C_{i,\pi(i)} \prod_{i\in V(K)} C_{i,\pi'(i)} \\
&= \operatorname{sgn}(\pi) \prod_{i\in\{1,\ldots,n\}\setminus V(K)} C_{i,\pi(i)} \prod_{i\in V(K)} C_{\pi(i),i} \\
&= (-1)^{|V(K)|} \cdot \operatorname{sgn}(\pi) \prod_{i\in\{1,\ldots,n\}\setminus V(K)} C_{i,\pi(i)} \prod_{i\in V(K)} C_{i,\pi(i)} \\
&= -\operatorname{sgn}(\pi) \prod_{i\in\{1,\ldots,n\}} C_{i,\pi(i)}
\end{aligned}$$

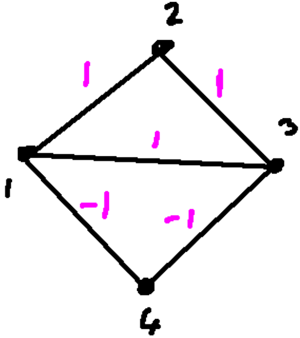
The antisymmetric adjacency matrix of (G, b):
$C_{u,v} = b(uv)\lambda(u, v)$ for $uv \in E(G)$, 0 otherwise.

Lemma

$$\det(C) = \mathrm{Pf}^2(G, b).$$

$$\begin{aligned}
\det(C) &= \sum_{\pi \text{ even-c.}} \mathrm{sgn}(\pi) \prod_{i=1}^{n} C_{i,\pi(i)} = \sum_{\pi \text{ even-c.}} \mathrm{sgn}(F_\pi)\lambda(F_\pi) \prod_{e \in F_\pi} b(e) \\
&= \sum_{M_1, M_2 \subseteq G \text{ perf. match.}} \mathrm{sgn}(M_1 + M_2)\lambda(M_1 + M_2) \prod_{e \in E(M_1 + M_2)} b(e) \\
&= \sum_{M_1, M_2 \subseteq G \text{ perf. match.}} \mathrm{sgn}(M_1)\mathrm{sgn}(M_2) \prod_{e \in E(M_1)} b(e) \prod_{e \in E(M_2)} b(e) \\
&= \Bigg(\sum_{M \subseteq G \text{ perfect matching}} \mathrm{sgn}(M) \prod_{e \in E(M)} b(e) \Bigg)^2 = \mathrm{Pf}^2(G, b).
\end{aligned}$$

Example:



$$C = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \det(C) = 4, \operatorname{Pf}(G, b) = -2.$$

Theorem (Kasteleyn)

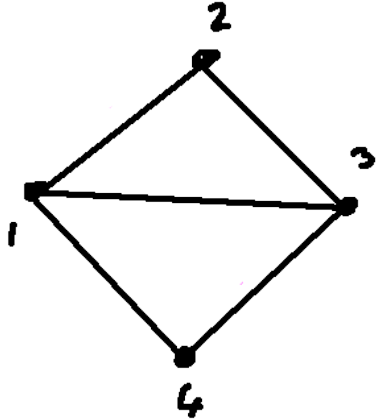
For every planar graph G, we can find a Pfaffian function b in polynomial time.

For a face f with boundary $v_1 v_2 \dots v_t$ in clockwise order, let

$$B(f) = \{(v_1, v_2), \dots, (v_{t-1}, v_t), (v_t, v_1)\}.$$

Choose b so that for every internal face f,

($\star$) the number of pairs $(u, v) \in B(f)$ such that $b(uv) \neq \lambda(u, v)$ is odd.



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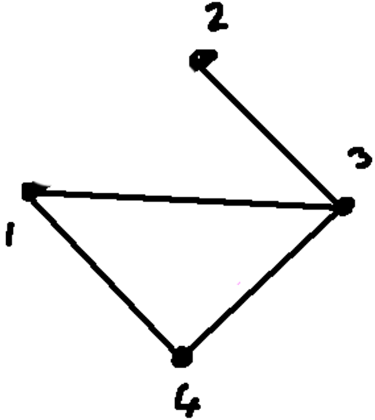

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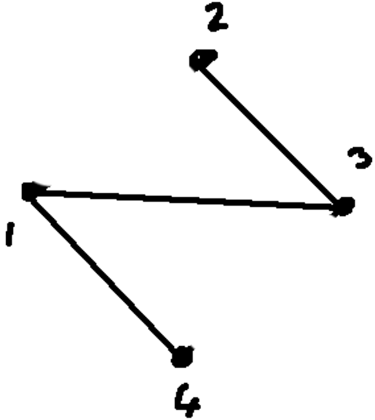

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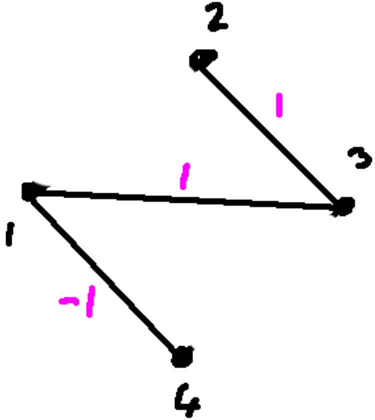

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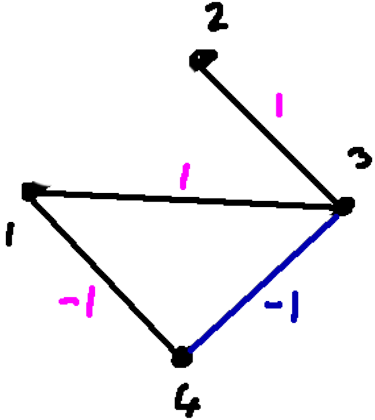

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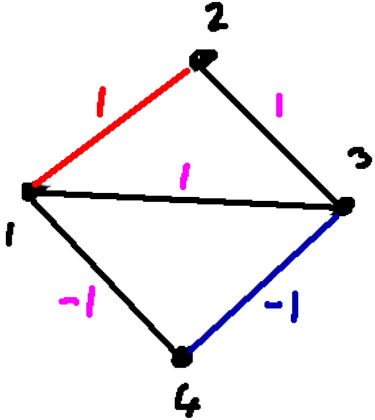
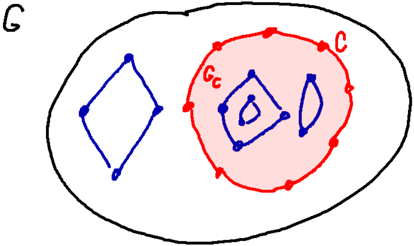
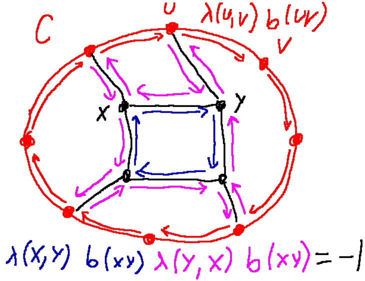


($\star$) for every internal face f, the number of pairs $(u, v) \in B(f)$ such that $b(uv) \neq \lambda(u, v)$ is odd.

Perf. matchings M_1, M_2: $F = M_1 + M_2$, cycle $C = v_1 \dots v_t$ of F:





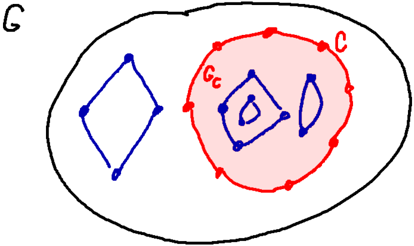
$$\lambda(C) \prod_{e \in E(C)} b(e) = \prod_{i=1}^{t} \lambda(u_i, u_{i+1}) b(u_i u_{i+1})$$

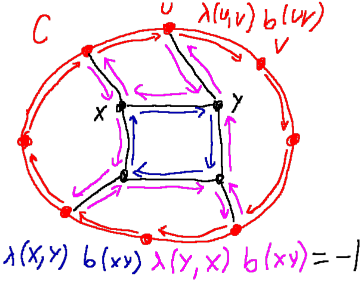
$$= (-1)^{|E(G_C)|-|E(C)|} \prod_{f \in F_i(G_C)} \prod_{(u,v) \in B(f)} \lambda(u, v) b(uv)$$

$$= (-1)^{|E(G_C)|-|E(C)|} \prod_{f \in F_i(G_C)} (-1) = -1.$$

$|E(G_C)| + 2 - (|F_i(G_C)| + 1) = |V(G_C)|$ is even.

Perf. matchings M_1, M_2: $F = M_1 + M_2$, cycle $C = v_1 \dots v_t$ of F:





$$\lambda(C) \prod_{e \in E(C)} b(e) = \prod_{i=1}^{t} \lambda(u_i, u_{i+1}) b(u_i u_{i+1})$$

$$= (-1)^{|E(G_C)| - |E(C)|} \prod_{f \in F_i(G_C)} \prod_{(u,v) \in B(f)} \lambda(u, v) b(uv)$$

$$= (-1)^{|E(G_C)| - |E(C)|} \prod_{f \in F_i(G_C)} (-1) = -1.$$

$$\begin{aligned}
&\Big(\mathrm{sgn}(M_1)\cdot \prod_{e\in E(M_1)} b(e)\Big)\cdot\Big(\mathrm{sgn}(M_2)\cdot \prod_{e\in E(M_2)} b(e)\Big)\\
&= \mathrm{sgn}(M_1)\mathrm{sgn}(M_2) \prod_{e\in E(F)} b(e)\\
&= \mathrm{sgn}(F)\lambda(F) \prod_{e\in E(F)} b(e)\\
&= \prod_{C \text{ cycle of } F}\Big((-1)\cdot\lambda(C)\prod_{e\in E(C)} b(e)\Big) = 1.
\end{aligned}$$

Hence, $\mathrm{sgn}(M)\cdot\prod_{e\in E(M)} b(e)$ is the same for every perfect matching M of G.