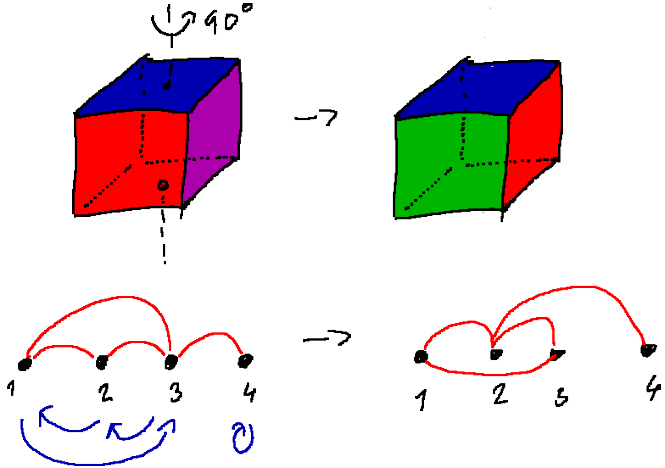
- # of ways how to color faces of a cube by k colors, when colorings differing only by a rotation are considered to be the same?
- # of pairwise non-isomorphic graphs on n vertices?



Counting objects from some set X

- k-colorings of cube faces
- graphs with vertex set $\{1, \ldots, n\}$

subject to some symmetries

- rotations
- permutations of vertices

Definition

A group G is a set with a binary associative operation $\circ$, the identity element 1, and the inverse x^{-1}:

- $x \circ (y \circ z) = (x \circ y) \circ z$
- $x \circ 1 = 1 \circ x = x$
- $x \circ x^{-1} = x^{-1} \circ x = 1$

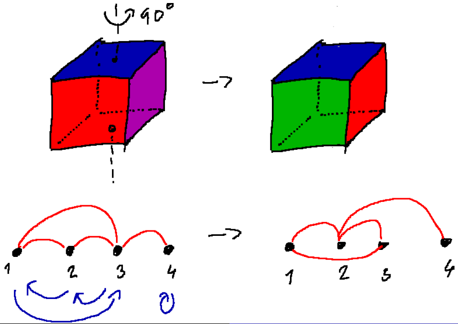
- R_{cube}: rotations that map the cube to itself, with composition of the rotations.
- Sym_n: all permutations of $\{1, \ldots, n\}$, with composition of the permutations.

Definition

An action of a group G on a set X: a function $\cdot : G \times X \to X$ s.t.

- $1 \cdot x = x$
- $(g \circ h) \cdot x = h \cdot (g \cdot x)$

- r rotation of a cube, x k-coloring: $r \cdot x = x$ rotated by r.
- π permutation of vertices of a graph H: $\pi \cdot H =$ graph with edges $\{\pi(u)\pi(v) : uv \in E(H)\}$.



Definition

An action of a group G on a set X: a function $\cdot : G \times X \to X$ s.t.

- $1 \cdot x = x$
- $(g \circ h) \cdot x = h \cdot (g \cdot x)$

Observation

For $a_g : X \to X$ defined by $a_g(x) = g \cdot x$:

- *a_g is a permutation of X*
- *$a_{g^{-1}}$ is the inverse permutation to a_g*

For a group G with an action $\cdot$ on X:

- $x \sim y$ if there exists $g \in G$ such that $g \cdot x = y$
- $\sim$ is an equivalence

Definition

The classes of equivalence of $\sim$ are orbits.

- colorings of cube faces obtainable from x by rotations
- graphs obtainable from H by permuting vertices

Problem

How many orbits does the action $\cdot$ have?

For a group G with action $\cdot$ on X:

- $\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$
- $\mathrm{Map}(x, y) = \{g \in G : g \cdot x = y\}$

Lemma

Let O be the orbit containing x:

- $G = \dot{\bigcup}_{y \in O} \mathrm{Map}(x, y)$.
- *If $h \cdot y = z$, then* $\mathrm{Map}(x, z) = \{g \circ h : g \in \mathrm{Map}(x, y)\}$.
- $|\mathrm{Map}(x, x)| = |G|/|O|$

Theorem (Burnside's lemma)

$$\text{\# of orbits} = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g)$$

Proof.

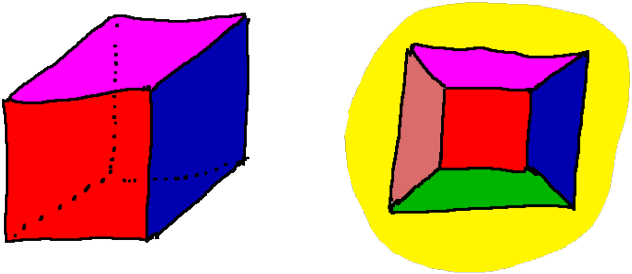
Let the orbits be $O_1, \ldots, O_m$:

$$\begin{aligned}
\sum_{g \in G} \mathrm{Fix}(g) = |\{(g,x) : g \cdot x = x\}| &= \sum_{x \in X} |\mathrm{Map}(x,x)| \\
&= \sum_{i=1}^{m} \sum_{x \in O_i} |\mathrm{Map}(x,x)| = \sum_{i=1}^{m} \sum_{x \in O_i} |G|/|O_i| \\
&= \sum_{i=1}^{m} |G| = |G|m
\end{aligned}$$

□

Cube rotations:

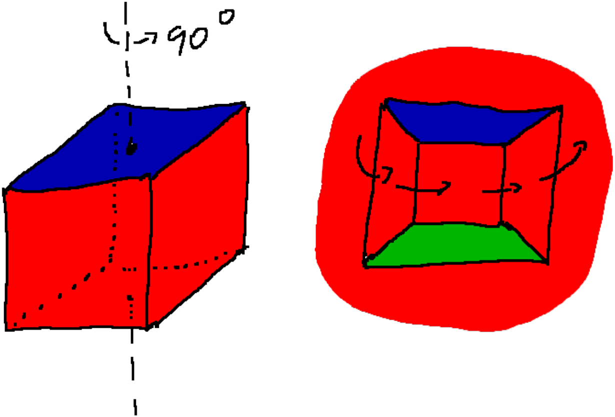
The identity: k^6 fixed points.



$$\frac{1}{24}\left(k^6 \qquad\qquad\qquad\qquad\qquad\qquad\right)$$

Cube rotations:

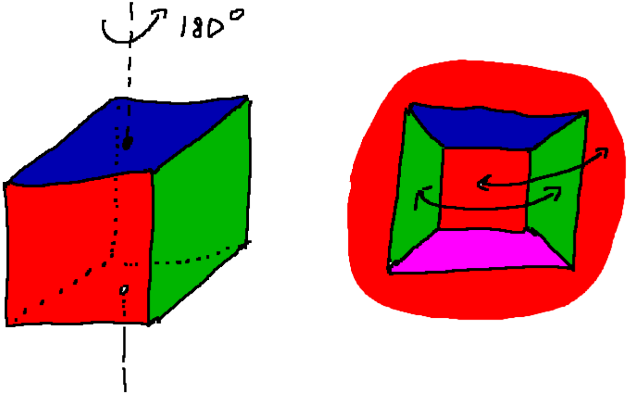
Axis through face centers, $\pm 90^\circ$: k^3 fixed points.



$$\frac{1}{24}\left(k^6 \qquad + 6k^3 \qquad\right)$$

Cube rotations:

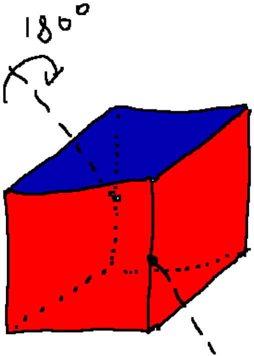
Axis through face centers, 180°: k^4 fixed points.

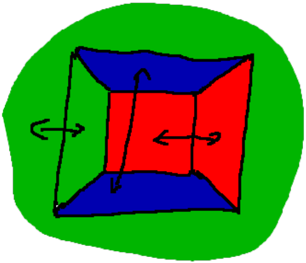


$$\frac{1}{24}\left(k^6 + 3k^4 + 6k^3 \qquad\qquad \right)$$

Cube rotations:

Axis through edge centers, 180°: k^3 fixed points.

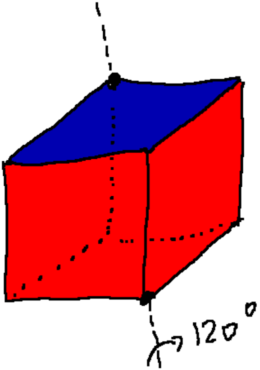


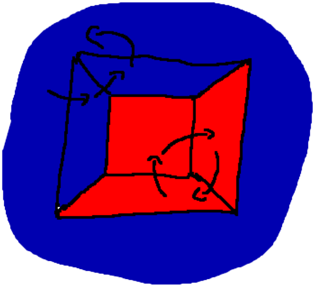


$$\frac{1}{24}\left(k^6 + 3k^4 + 12k^3 \qquad\qquad \right)$$

Cube rotations:

Axis through vertices, $\pm 120^\circ$: k^2 fixed points.

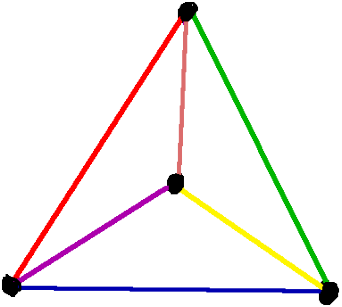




$$\frac{1}{24}\left(k^6 + 3k^4 + 12k^3 + 8k^2\right)$$

Vertex permutations:

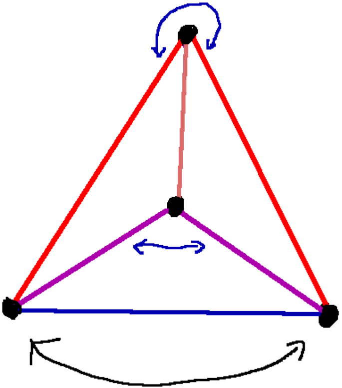
The identity: 2^6 fixed points.



$$\frac{1}{24}\left(2^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\right)$$

Vertex permutations:

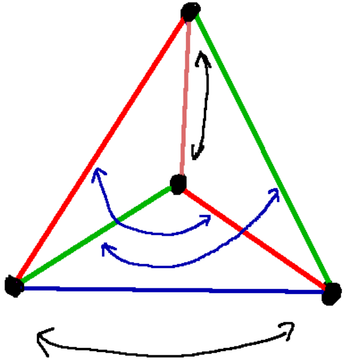
(ab): 2^4 fixed points.



$$\frac{1}{24}\left(2^6 + 6 \cdot 2^4 \qquad\qquad\qquad\qquad\qquad\qquad \right)$$

Vertex permutations:

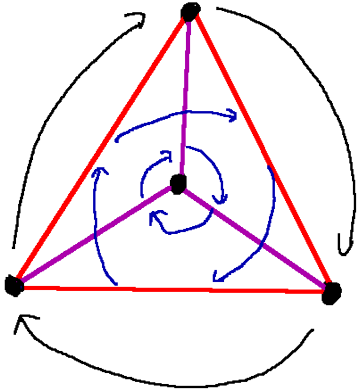
$(ab)(cd)$: 2^4 fixed points.



$$\frac{1}{24}\left(2^6 + 6 \cdot 2^4 + 3 \cdot 2^4 \qquad\qquad\qquad\qquad \right)$$

Vertex permutations:

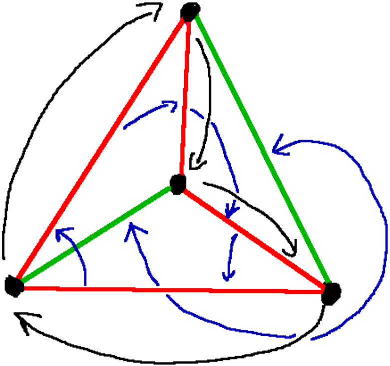
(abc): 2^2 fixed points.



$$\frac{1}{24}\left(2^6 + 6 \cdot 2^4 + 3 \cdot 2^4 + 8 \cdot 2^2 \qquad\qquad \right)$$

Vertex permutations:

$(abcd)$: 2^2 fixed points.



$$\frac{1}{24}\left(2^6 + 6 \cdot 2^4 + 3 \cdot 2^4 + 8 \cdot 2^2 + 6 \cdot 2^2\right) = 11$$

Is it faster than a direct enumeration?

- $2^{\binom{n}{2}} = 2^{\Theta(n^2)}$ graphs.
- $> 2^{\binom{n}{2}}/n! = 2^{\Theta(n^2)}/2^{\Theta(n \log n)} = 2^{\Theta(n^2)}$ non-isomorphic graphs.
- $n! = 2^{\Theta(n \log n)}$ permutations.
- $2^{\Theta(\sqrt{n})}$ cycle structures.

Pólya enumeration:

- k boxes, G: a group of some of their permutations
- For $i = 0, 1, 2, \ldots$: a_i kinds of objects of size i

Problem (Pólya enumeration problem)

# of ways to put an object to each box s.t. the total size is m:

- *the same kind can be used multiple times*
- *arrangements differing only by permutations in G are considered to be the same.*

For a permutation π:

- $c_\ell(\pi) =$ number of cycles of π of length ℓ.

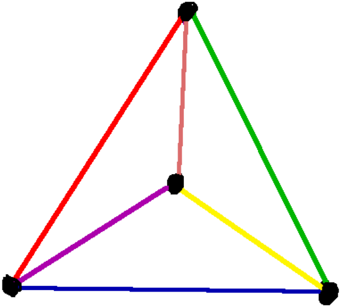
Definition

The cycle index of a group G of permutations is

$$Z_G(x_1, x_2, \ldots) = \frac{1}{|G|} \sum_{\pi \in G} \prod_{\ell \geq 1} x_\ell^{c_\ell(\pi)}.$$

Boxes=pairs, $G =$ their permutations induced by vertex permutations.

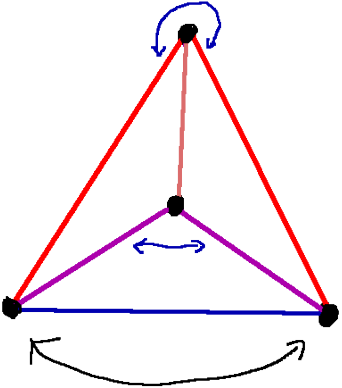
The identity: $6 \times$ 1-cycle



$$\frac{1}{24}\left(x_1^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\right).$$

Boxes=pairs, $G =$ their permutations induced by vertex permutations.

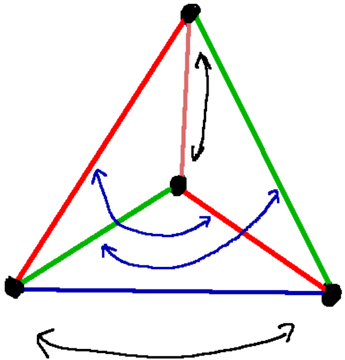
(ab): $2 \times$ 1-cycle, $2 \times$ 2-cycle



$$\frac{1}{24}\left(x_1^6 + 6x_1^2x_2^2 \qquad\qquad\qquad\qquad \right).$$

Boxes=pairs, $G =$ their permutations induced by vertex permutations.

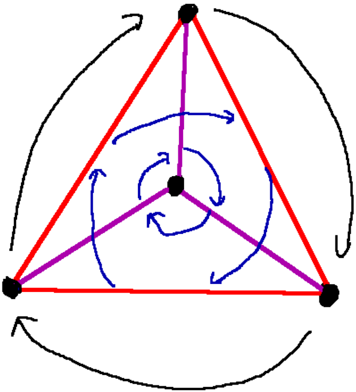
$(ab)(cd)$: $2 \times$ 1-cycle, $2 \times$ 2-cycle



$$\frac{1}{24}\left(x_1^6 + 9x_1^2x_2^2 \qquad\qquad\qquad \right).$$

Boxes=pairs, $G =$ their permutations induced by vertex permutations.

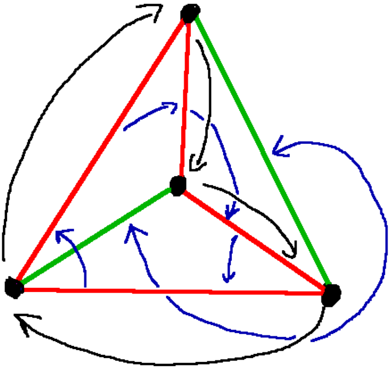
(abc): $2 \times$ 3-cycle



$$\frac{1}{24}\left(x_1^6 + 9x_1^2x_2^2 + 8x_3^2 \qquad\qquad\right).$$

Boxes=pairs, $G =$ their permutations induced by vertex permutations.

$(abcd)$: 2-cycle and 4-cycle



$$\frac{1}{24}\left(x_1^6 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4\right).$$

- k boxes, G: a group of some of their permutations
- For $i = 0, 1, 2, \ldots$: a_i kinds of objects of size i
- # of arrangements of total size m, up to symmetries given by G

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

Generating function for choosing ℓ copies of the same kind:

$$a_0 + a_1 x^\ell + a_2 x^{2\ell} + \ldots = A(x^\ell)$$

Arrangements fixed by π: Placing copies of the same kind in each cycle.

$$|\mathrm{Fix}(\pi)| = [x^m] \prod_{\ell \geq 1} A^{c_\ell(\pi)}(x^\ell)$$

The answer to Pólya enumeration problem is

$$\frac{1}{|G|}\sum_{\pi\in G}[x^m]\prod_{\ell\geq 1}A^{c_\ell(\pi)}(x^\ell) = [x^m]Z_G(A(x), A(x^2), A(x^3), \ldots).$$

Number of non-isomorphic graphs with 4 vertices and m edges:

- $a_0 = 1$ (non-edge), $a_1 = 1$ (edge), $A(x) = 1 + x$.

$$
\begin{aligned}
&Z_G(1+x, 1+x^2, \ldots) \\
&\quad = \frac{1}{24}\big((1+x)^6 + 9(1+x)^2(1+x^2)^2 + 8(1+x^3)^2 \\
&\qquad\qquad + 6(1+x^2)(1+x^4)\big) \\
&\quad = \frac{1}{24}\left(24 + 24x + 48x^2 + 72x^3 + 48x^4 + 24x^5 + 24x^6\right) \\
&\quad = 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6
\end{aligned}
$$