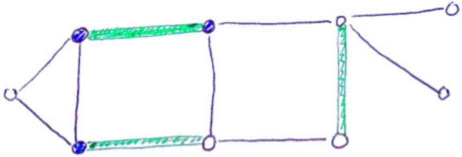
- matching: 1-regular subgraph M
- size: $|E(M)|$; $\beta(G) =$ size of a largest matching
- covers X if $X \subseteq V(M)$; perfect if $V(M) = V(G)$.



size of $M = 3$

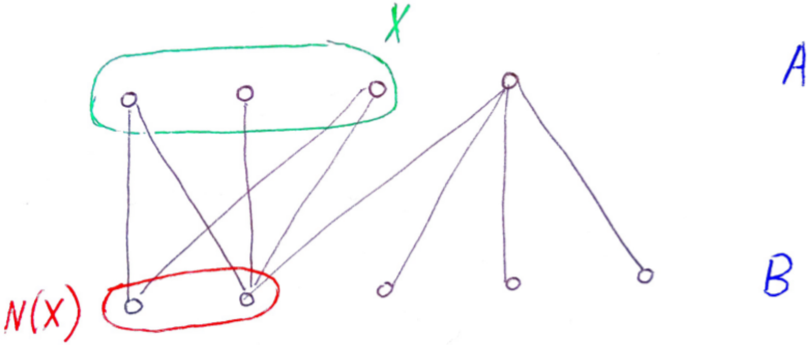
M covers X

Q: What is β(the depicted graph)?

Hall's theorem

G bipartite, parts A and B. Equivalent:

- Exists a matching that covers A.
- For every $X \subseteq A$,
$$|N(X)| \geq |X|.$$



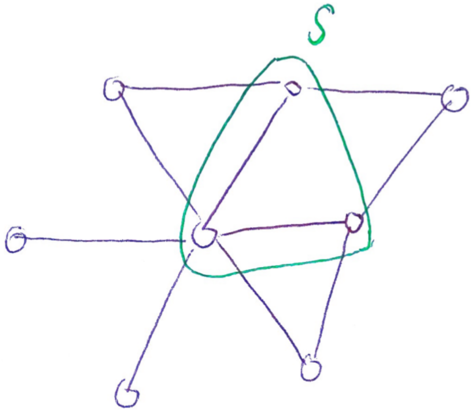
Corollary

G bipartite and d-regular $\Rightarrow$ G has a perfect matching.

$$d|X| = |E(G[X \cup N(X)])| \leq d|N(X)|$$

Observation

If $V(G) \setminus S$ is an independent set, then $\beta(G) \leq |S|$.
Hence, $\beta(G) \leq |V(G)| - \alpha(G)$.



Theorem

For G bipartite:

There exists $S \subseteq V(G)$ such that

- *$V(G) \setminus S$ is an independent set and*
- *$\beta(G) = |S|$.*

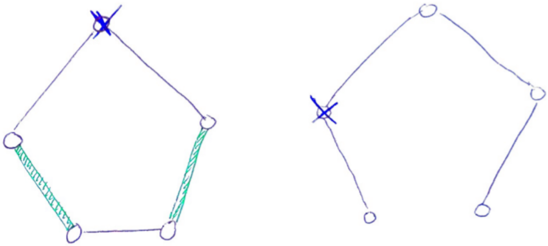
Hence, $\beta(G) = |V(G)| - \alpha(G)$.

Q: Find a non-bipartite graph G such that

$$\beta(G) < |V(G)| - \alpha(G).$$

A graph G is hypomatchable if

- G does not have a perfect matching
- for every $v \in V(G)$, $G - v$ has a perfect matching.



Observation

A hypomatchable graph must have an odd number of vertices.

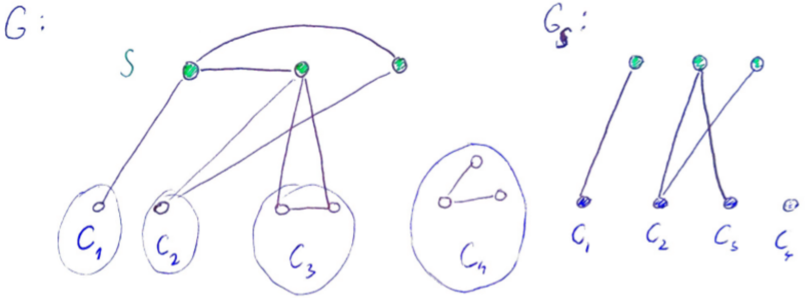
Q: Find a hypomatchable graph with 5 vertices which does not contain a 5-cycle.

For a graph G and $S \subseteq V(G)$, G_S is the bipartite graph with parts

- S and
- components of $G - S$

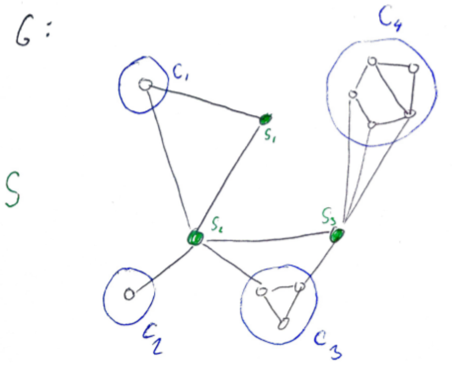
and for $v \in S$ and a component C of $G - S$,

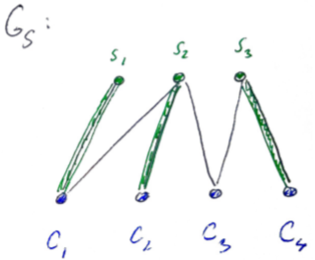
- $vC \in E(G_S)$ iff G has an edge from v to $V(C)$.



A set $S \subseteq V(G)$ is an EG-set (Edmonds-Gallai) if

- every component of $G - S$ is hypomatchable, and
- G_S has a matching that covers S.



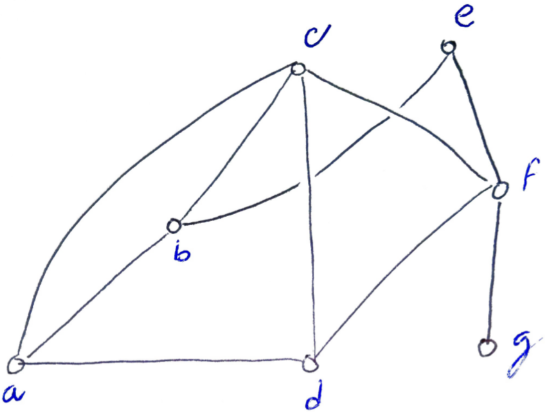


Observation

$\emptyset$ is an EG-set iff G is hypomatchable.

A set $S \subseteq V(G)$ is an EG-set (Edmonds-Gallai) if

- every component of $G - S$ is hypomatchable, and
- G_S has a matching that covers S.



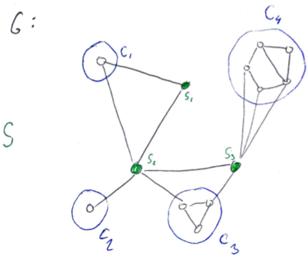
Q: Find an EG-set in the graph above.

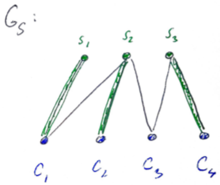
Lemma

If S is an EG-set and $G - S$ has c components, then

$$\beta(G) = \tfrac{1}{2}(|V(G)| + |S| - c).$$

- In any matching, at least $c - |S|$ components contain an uncovered vertex.
- There exists a matching with exactly $c - |S|$ uncovered vertices.





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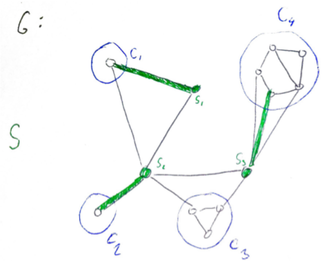

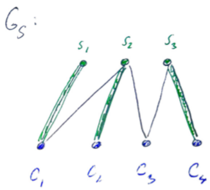

Lemma

If S is an EG-set and G − S has c components, then

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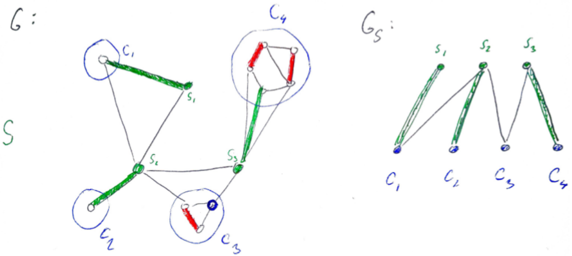
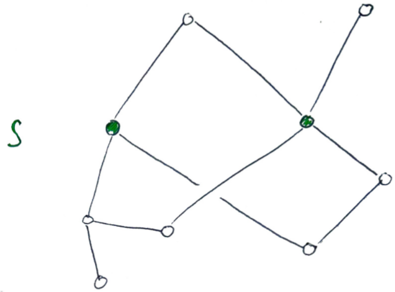


Theorem (Edmonds-Gallai)

Every graph contains an EG-set.

We will prove this Theorem at the end of the lecture.

$o(H)$ = number of odd-size components of H



Q: What is $o(G - S)$ for the graph above?

Observation

If G has a perfect matching, then for every $S \subseteq V(G)$,

$$o(G - S) \leq |S|.$$

Theorem (Tutte)

G has a perfect matching iff for every $S \subseteq V(G)$,

$$o(G - S) \leq |S|.$$

Proof.

- By Edmonds-Gallai Theorem, G contains an EG-set S.
- All components of $G - S$ have odd size, there are $o(G - S)$ of them.
- $\beta(G) = \frac{1}{2}(|V(G)| + |S| - o(G - S)) \geq \frac{1}{2}|V(G)|$.

$\square$

Theorem (Petersen)

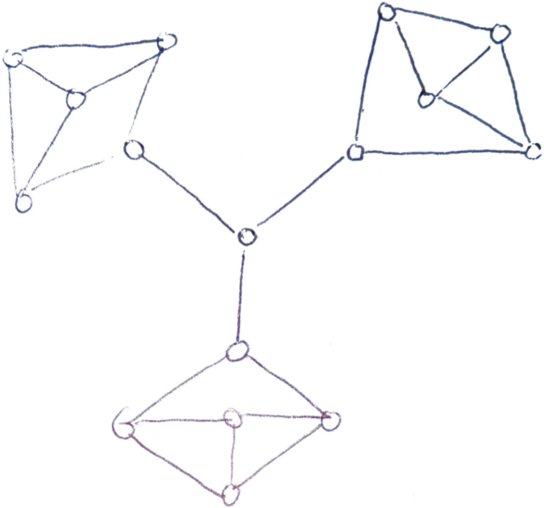
Every 3-regular 2-edge-connected graph has a perfect matching.

Proof.

- For every odd-size component C of $G - S$, the number of edges between S and C is odd.
- If $S \neq \emptyset$, it is at least three, since G is 2-edge-connected.
- $3o(G - S) \leq$ edges between S and $V(G) \setminus S \leq 3|S|$.
- $o(G - \emptyset) = 0$.

$\square$

Q: Is it true that every 3-regular graph has a perfect matching?



Theorem (Edmonds-Gallai)

Every graph G contains an EG-set.

Induction: We can assume every graph with less than $|V(G)|$ vertices has an EG-set.

Choose $S \subseteq V(G)$ such that

- $o(G - S) - |S|$ is maximum, and subject to that
- $|S|$ is maximum

Then S is an EG-set.

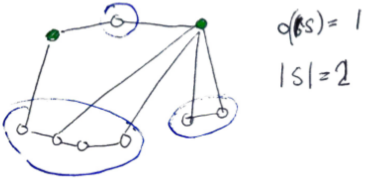
Claim

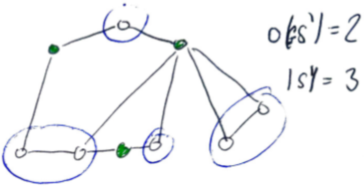
Every component C of $G - S$ has odd size.

Otherwise:

- Choose $v \in V(C)$ arbitrarily, let $S' = S \cup \{v\}$.
- $C - v$ has an odd component: $o(G - S') \geq o(G - S) + 1$.

$o(G - S') - |S'| \geq (o(G - S) + 1) - (|S| + 1) = o(G - S) - |S|$ and $|S'| > |S|$.





Claim

Every component C of $G - S$ is hypomatchable.

Otherwise:

- Choose $v \in V(C)$ such that $\beta(C - v) < |V(C - v)|/2$.
- Hence, $\beta(C - v) \leq |V(C - v)|/2 - 1 = (|V(C - v)| - 2)/2$.
- EG-set S_C in $C - v$: $o(C - v - S_C) - |S_C| \geq 2$.
- For $S' = S \cup \{v\} \cup S_C$,
  $o(G - S') = (o(G - S) - 1) + o(C - v - S_C)$.

$$\begin{aligned} o(G - S') - |S'| &= (o(G - S) + o(C - v - S_C) - 1) - (|S| + |S_C| + 1) \\ &= o(G - S) - |S| + (o(C - v - S_C) - |S_C| - 2) \\ &\geq o(G - S) - |S| \end{aligned}$$

and $|S'| > |S|$.

Claim

G_S has a matching that covers S.

Otherwise:

- Hall's theorem: $X \subseteq G$ such that $|N_{G_S}(X)| < |X|$.
- For $S' = S \setminus X$,
  $o(G - S') \geq o(G - S) - |N_{G_S}(X)| > o(G - S) - |X|$.

$$o(G - S') - |S'| > (o(G - S) - |X|) - (|S| - |X|) = o(G - S) - |S|.$$