Graph coloring: Heawood and Brooks theorems, edge coloring

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1 Heawood's formula

What can we say about the chromatic number of graphs drawn in a given surface? For a surface Σ, let $\chi(\Sigma)$ denote the maximum chromatic number of a graph drawn in Σ. So, for example, the Four Color Theorem shows that $\chi(\text{sphere}) = 4$. Let us also define $\omega(\Sigma)$ as the order of the largest clique that can be drawn in Σ. Clearly, $\chi(\Sigma) \geq \omega(\Sigma)$.

Observe that Hadwiger's conjecture implies that actually $\chi(\Sigma) = \omega(\Sigma)$. Indeed, consider any graph G drawn in Σ. If Hadwiger's conjecture is true, then G contains $K_{\chi(G)}$ as a minor. However, any minor of G is also drawn in Σ, and thus the order $\chi(G)$ of this clique must be at most $\omega(\Sigma)$.

In fact, this result has been proven unconditionally, without needing to assume the validity of Hadwiger's conjecture. Let us define

$$H(g) = \left\lceil \frac{7 + \sqrt{24g + 1}}{2} \right\rceil.$$

Theorem 1 (Heawood's formula)**.** *Every graph drawn in a surface of Euler genus g has chromatic number at most $H(g)$.*

Proof. Note that $H(0) = 4$ and the graphs drawn in the sphere are 4-colorable by the Four Color Theorem. Hence, we can assume $g > 0$.

Suppose for a contradiction that there exists a graph drawn in a surface of Euler genus g with chromatic number is greater than $H(g)$, and let G be such a graph with the smallest number n of vertices. Clearly $n \geq H(g) + 1$. Moreover, G has minimum degree at least $H(g)$: If G contained a vertex v of smaller degree, we could $H(g)$-color $G - v$ by the minimality of G, then extend the coloring to G by giving v a color different from the colors of its neighbors.

On the other hand, as we learned the last time, G has at most $3n+3g-6$ edges, and thus its average degree is at most $6 + \frac{6(g-2)}{n}$. For $g = 1$, this is a contradiction, as this shows that the average degree is less than 6, while in the previous paragraph, we argued the minimum degree is at least $H(1) = 6$. Hence, assume $g \geq 2$.

Then, since $n \geq H(g) + 1$, we conclude that the average degree of G is at most $6 + \frac{6(g-2)}{n} \leq 6 + \frac{6(g-2)}{H(g)+1}$, and comparing the average degree with the minimum degree, we conclude

$$\begin{aligned} H(g) &\leq 6 + \tfrac{6(g-2)}{H(g)+1} H^2(g) - 5H(g) - 6(g-1) && \leq 0 \\ H(g) &\leq \frac{5 + \sqrt{24g+1}}{2}, \end{aligned}$$

contradicting the definition of $H(g)$. $\square$

On the other hand, Ringel and Youngs proved the following (much harder) result.

Theorem 2. *If Σ is a surface of Euler genus g and Σ is not the Klein bottle, then $K_{H(g)}$ can be drawn in Σ.*

We can now combining this theorem with Heawood's formula.

Corollary 3. *For every surface Σ other than the Klein bottle, denoting by g the Euler genus of Σ, we have $\chi(\Sigma) = \omega(\Sigma) = H(g)$.*

What about the Klein bottle?

Exercise 4. *Show that it is possible to draw K_6 but not K_7 in the Klein bottle, and thus ω(Klein bottle) $= 6$.*

We will prove that also χ(Klein bottle) $= 6$. First, let us see how the argument used to prove Heawood's formula constrains a hypothetical minimal counterexample.

Lemma 5. *If G is a non-6-colorable graph drawn in the Klein bottle with the smallest number of vertices, then G is 6-regular.*

Proof. As in the proof of Heawood's formula, we argue G has minimum degree at least 6. Moreover, since the Klein bottle has Euler genus 2, we have $|E(G)| \leq 3|V(G)|$ from the generalized Euler formula, and thus G has average degree at most 6. This is only possible if G is 6-regular. $\square$

The main result of the following section will finish the argument.

2 Brooks theorem

A straightforward greedy coloring argument shows that every graph G can be colored by $\Delta(G)+1$ colors. Indeed, if the graph is not regular, we can save a color.

Lemma 6. *Let G be a graph of maximum degree at most Δ. If every component of G contains a vertex of degree less than Δ, then $\chi(G) \le \Delta$.*

Proof. We prove the claim by induction on $|V(G)|$, with the basic case $|V(G)| = 1$ being trivial. By the assumptions, G contains a vertex v of degree less than Δ. Note that for every component C of $G-v$, either C is also a component of G, or C contains a neighbor of v, and thus C contans a vertex of degree less than Δ. By the induction hypothesis, $G-v$ can be colored by Δ colors. Since v has degree less than Δ, we can extend this coloring to a coloring of G by giving v a color different from the colors of all its neighbors. $\square$

Moreover, we can extend this result to graphs that are Δ-regular, but not 3-connected.

Lemma 7. *Let $\Delta \ge 3$ be an integer and let G be a connected graph of maximum degree at most Δ. If G is not 3-connected, then $\chi(G) \le \Delta$.*

Proof. Let S be a minimum cut in G and let $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 intersecting only in S. If $S = \{v\}$, then note that v has degree less than Δ both in G_1 and in G_2, and thus both of these graphs can be Δ-colored by Lemma 6. Moreover, without loss of generality, we can assume v gets color 1 in both colorings. Then the union of the colorings is a Δ-coloring of G.

Hence, suppose $S = \{u, v\}$. Without loss of generality, we can assume that u has at least two neighbors in G_1 (as otherwise we could consider the cut consisting of v and the neighbor of u in G_1, instead), and similarly, that v has at least two neighbors in G_2. Then u has degree less than Δ in $G_2 + uv$ and v has degree less than Δ in $G_1 + uv$, and thus these graphs have a Δ-coloring by Lemma 6. In both colorings, we can without loss of generality assume u has color 1 and v has color 2, and thus they combine to a Δ-coloring of G. $\square$

In fact, much more is true: The only connected graphs for which Δ colors do not suffice are cliques and odd cycles!

Lemma 8. *Let $\Delta \ge 3$ be an integer and let G be a 3-connected graph of maximum degree at most Δ. If $G \ne K_{\Delta+1}$, then $\chi(G) \le \Delta$.*

Proof. Since G is not a clique, it contains non-adjacent vertices. Choose such vertices u and v at distance two from one another, and let w be a common neighbor of u and v. Since G is 3-connected, $G - \{u, v\}$ is connected. Let v_1, …, v_n be an ordering of vertices of G such that $v_1 = u$, $v_2 = v$, $v_n = w$, and $v_3, \ldots, v_{n-1}$ are sorted in a non-increasing order according to their distance from w in $G - \{u, v\}$. Let us give u and v the color 1.

Next, we color v_3, …, v_{n-1} in order. For $i = 3, \ldots, n-1$, note that v_i has a neighbor v_j on a shortest path from v_i to w in $G - \{u, v\}$, and by the choice of the ordering, we have $j > i$. Hence, when we are coloring v_i, the vertex v_j is still uncolored, and thus v_i has at most $\Delta - 1$ neighbors to which we already assigned a color. Hence, we can color v_i by a color from $\{1, \ldots, \Delta\}$ different from the colors of these neighbors.

Finally, we color w; since both u and v are neighbors of w and they received the same color, there are again at most $\Delta - 1$ distinct colors appearing on the neighborhood of w, and thus we can give w a different color from $\{1, \ldots, \Delta\}$. □

Let us now put all of these partial results together.

Theorem 9 (Brooks)**.** *Let G be a connected graph of maximum degree Δ. If G is neither a clique nor an odd cycle, then $\chi(G) \leq \Delta$.*

Proof. We cannot have $\Delta = 1$, since G is not the clique K_2. For $\Delta = 2$, G is a path or an even cycle, and thus G is 2-colorable. Hence, suppose $\Delta \geq 3$. Then G is Δ-colorable by Lemma 7 or Lemma 8, depending on whether G is 3-connected. □

Corollary 10. *Every graph G drawn in the Klein bottle is* 6*-colorable.*

Proof. By Lemma 5, we can assume G is 6-regular, and by Lemma 4 we have $G \neq K_7$. Hence, G is 6-colorable by Theorem 9. □

Exercise 11. *Prove that a graph G drawn in the torus is* 6*-colorable if and only if $\omega(G) \leq 6$.*

3 Edge coloring

Instead of giving colors to vertices, we can assign them to edges.

Definition 12. *A function $\varphi : E(G) \to \{1, \ldots, k\}$ is an* edge k-coloring *of G if for every distinct edges $e_1, e_2 \in E(G)$ incident with the same vertex, we have $\varphi(e_1) \neq \varphi(e_2)$.*

The *chromatic index* $\chi'(G)$ of G is the minimum k such that G has an edge k-coloring.

For example, suppose you are organizing a tournament and some of the teams already played against one another. You are given a graph G in which the pairs of teams that still need to play a game are joined by an edge. In each round, there can be any number of matches running in parallel, but of course every team can play in at most one of them. How many rounds are at minimum needed to finish the tournament? Coloring by i the edges corresponding to the matches played in the i-th round, we conclude we need precisely $\chi'(G)$ rounds.

Observation 13. *In an edge coloring, edges incident with the same vertex have pairwise distinct colors, and thus $\chi'(G) \geq \Delta(G)$. Edges of the same color form a matching, and thus if the largest matching in G has b edges, then $\chi'(G) \geq |E(G)|/b$.*

Exercise 14. *Show that*

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

We can translate edge coloring to vertex coloring. The *linegraph* $L(G)$ of a graph G is the graph with vertex set $E(G)$ and such that distinct $e_1, e_2 \in E(G)$ are adjacent if and only if e_1 and e_2 are incident with the same vertex.

Observation 15. $\chi'(G) = \chi(L(G))$

However, not every graph is a linegraph, and thus we may be able to obtain results for edge coloring that do not hold for ordinary coloring.

Exercise 16. *Show that there does not exist any graph G such that $L(G) = K_{1,3}$.*

Note that $L(G)$ has maximum degree at most $2\Delta(G) - 2$.

Observation 17. $\chi'(G) \leq 2\Delta(G) - 1$

Exercise 18. *Using Brook's theorem, we can improve this bound to $2\Delta(G) - 2$, with some exceptions. What are the exceptions?*

However, much more is true in general.

Theorem 19 (Vizing)**.** *Any simple graph G satisfies $\chi'(G) \leq \Delta(G) + 1$.*

Note that this is not true for multigraphs (the triangle T_k with each edge of multiplicity k has $\Delta(T_k) = 2k$ and $\chi'(T_k) = 3k$). By Observation 13, we have $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ for every G. However, in general it is hard to determine $\chi'(G)$ exactly; for example, it is NP-hard to distinguish whether a 3-regular graph has chromatic index 3 or 4.

For the proof of Vizing's theorem, we use the notion of *Kempe chains.* Consider a proper edge coloring of a graph G. A *Kempe chain* in colors $\{a, b\}$ is a maximal connected subgraph K of G such that every edge of H is colored a or b. Note that since each vertex is incident with at most one edge of each color, K has maximum degree at most two, and thus K is either a path or a cycle. Moreover, the colors a and b alternate on K, and thus if K is a cycle, then K has even length. By *switching* the Kempe chain K, we mean changing the color of each edge of K to the other color in $\{a, b\}$. Note this results in another proper coloring of edges of G.

Exercise 20. *Show that if G is a d-regular graph of chromatic index d and G has (up to renaming of the colors) only one proper edge coloring using d colors, then G contains a Hamilton cycle (i.e., a cycle containing all the vertices of G).*

Given a graph G with a proper coloring of its edges, we say that a color c is *missing* at a vertex v of G if no edge incident with v has color c. Note that if G is colored using more than $\Delta(G)$ colors, then at least one color is missing at each vertex of G. Consider a Kempe chain K in colors $\{a, b\}$. If K is a cycle, then switching K does not change which colors are missing at any vertex. If K is a path, then one of the colors $\{a, b\}$ must be missing at each end of K (by the maximality of the chain) and switching K makes the other color in $\{a, b\}$ missing at each end. We now prove the key lemma towards the proof of Vizing's theorem.

Lemma 21. *Let G be a graph of maximum degree at most Δ with edges properly colored using $\Delta + 1$ colors. Let u and v be distinct non-adjacent vertices of G. Then there exists a proper edge coloring using $\Delta + 1$ colors such that the same color missing at both u and v.*

Proof. Let c_1 be a color missing at u. Without loss of generality, we can assume that c_1 is not missing at v; let $e_1 = vx_1$ be an edge of color c_1 incident with v. We now repeat the following procedure for $i = 1, 2, \ldots$, as long as possible, building a "fan" of edges incident with v: Choose a color $c_{i+1} \notin \{c_1, \ldots, c_i\}$ missing at x_i but not at v, and let $e_{i+1} = vx_{i+1}$ be an edge of color c_{i+1} incident with v.

As the number of colors is bounded, this procedure eventually stops; let c_k and $e_k = vx_k$ be the last chosen color and edge. There is some color a missing at x_k, but this color cannot be chosen to be c_{k+1}; there are two possibilities why this might be the case. It could be that the color a is also missing at v. Hence, we can change the color of e_k to a. In the resulting coloring, the color c_k is missing at v, and thus we can change the color of e_{k-1} to c_k. We continue this process, recoloring e_{i-1} by the color c_i for $i = k, k-1, \ldots, 2$. At the end, the color c_1 is missing at v. It is also still missing at u, since u is not adjacent to v, and thus it is different from the vertices $x_1, \ldots, x_k$ for which the set of missing colors changed.

Hence, we can assume that the color a is not missing at v, but it is equal to c_s for some $s \in \{1, \ldots, k-1\}$. Let b be a color missing at v, and let K be the Kempe chain in colors $\{a, b\}$ containing x_k. Since a is missing at x_k, K is a path starting in x_k; let z denote the other end of the path K. Let us distinguish several cases based on where K ends.

- Suppose first $z \neq v$. Since b is missing at v, it follows that v is not an internal vertex of K, and thus $v \notin V(K)$. This implies that the edge e_s of color a is not contained in K, and thus $x_s \notin V(K)$.

  If $z \notin \{u, x_{s-1}\}$, then we switch the chain K. This gives an edge coloring in which the color b is missing at x_k, and moreover, edges incident with v have the same color, the color c_1 is still missing at u, and for $i = 1, \ldots, k-1$, the color c_{i+1} is missing at x_i. Hence, we can change the color of e_k to b, and for $i = k, k-1, \ldots, 2$, change the color of e_{i-1} to c_i; the color c_1 is missing at both u and v in the resulting coloring.

  Similarly, if $z = x_{s-1}$, we switch the chain K, ensuring the color b is missing at x_{s-1}, change the color of e_{s-1} to b, and for $i = s-1, \ldots, 2$, change the color of e_{i-1} to c_i, so that c_1 is missing at both u and v in the resulting coloring.

  Hence, suppose that $z = u$. If the color b is missing at u, then b is missing at both u and v and we are done. Otherwise, the edge of K incident with u has color b; switching the chain K, we obtain an edge coloring in which the color b is missing at both u and v.

- Finally, let us consider te case $z = v$. The edge e_s has color $c_s = a$, and thus e_s is the last edge of K. We switch K, obtaining an edge coloring such that b is missing at x_k, c_s is missing at v, e_s has color b, and moreover, no other edge incident with v changed color, c_1 is missing at u, and for $i = 1, \ldots, k-1$, the color c_{i+1} is missing at x_i. Now for $i = s, s-1, \ldots, 2$, change the color of e_{i-1} to c_i, so that the color c_1 is missing at both u and v in the resulting coloring.

$\square$

Vizing's theorem is now easy to prove.

Proof of Theorem 19. We prove the claim by inducion on $|E(G)|$; if $E(G) = \emptyset$, then the claim is trivial. Hence, suppose that there exists an edge $e = uv \in E(G)$. By the induction hypothesis, $G - e$ has an edge coloring using $\Delta(G') + 1 \leq \Delta(G) + 1$ colors. By Lemma 21, $G - e$ has such a coloring in which the same color c is missing at both u and v. We can color e by c to obtain an edge coloring of G using $\Delta(G) + 1$ colors. $\square$