The proof of Tutte's and Wagner's theorems; Hadwiger's conjecture

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1 Tutte's theorem

At the end of the previous lecture, we gave an inductive characterization of 3-connected graphs, which we prove now.

Theorem 1 (Tutte)**.** *For every 3-connected graph $G \neq K_4$, there exists an edge $e \in E(G)$ such that G/e is 3-connected.*

Proof. Suppose for a contradiction this is not the case, and thus for every $e \in E(G)$, the graph G/e contains a cut S'_e of size at most two. Let w_e be the vertex of G/e created by identifying the ends of $e = uv$. If $w_e \notin S'_e$, then S'_e would be a cut of size at most two in G, which is not possible. Hence, $w_e \in S'_e$. Let $S_e = (S'_e \setminus w_e) \cup \{u, v\}$. Then S_e is a cut in G, and thus $|S_e| \geq 3$. Since $|S'_e| \leq 2$ and $|S_e| = |S'_e| + 1$, we have $|S_e| = 3$. Therefore,

($\star$) for every $e = uv \in E(G)$, there exists a 3-cut $S_e \subseteq V(G)$ such that $u, v \in S_e$.

Let us consider an edge $e = uv \in E(G)$ and a component A of $G - S_e$ chosen so that (among all choices of edges and components) $|V(A)|$ is minimum. Let w be the vertex of S_e distinct from u and v. Since G is 3-connected, every vertex of S_e has a neighbor in A; let $e' = zw$ be an edge of G with $z \in V(A)$. Note that since uv is an edge, $\{u, v\}$ intersects only one component of $G - S_{e'}$. Let B be a component of $G - S_{e'}$ disjoint from $\{u, v\}$. In particular, $B \cap S_e = \emptyset$.

Since G is 3-connected, every vertex of $S_{e'}$ has a neighbor in B, and in particular, z has a neighbor $x \in V(B)$. Since $z \in V(A)$, all neighbors of z in G are contained in $V(A) \cup S_e$, and since $B \cap S_e = \emptyset$, it follows that $x \in V(A)$. Since A is a component of $G - S_e$, B is a connected subgraph of $G - S_e$, and $V(A) \cap V(B) \supseteq \{x\} \neq \emptyset$, it follows that $V(B) \subseteq V(A)$. Moreover, $z \in V(A) \setminus V(B)$, and thus $|V(B)| < |V(A)|$. This contradicts the minimality of A. $\square$

We say that a graph G is obtained from a graph G' by *splitting a vertex* if $G' = G/e$ for some $e \in E(G)$; equivalently, G is obtained from G' by selecting a vertex w and replacing it by adjacent vertices u and v whose neighborhoods are chosen so that $N_G(u) \cup N_G(v) = N_{G'}(w) \cup \{u, v\}$.

Exercise 2. *If G' is 3-connected, G is obtained from G' by splitting a vertex into an edge uv, and $\deg_G(u), \deg_G(v) \geq 3$, then G is 3-connected.*

Corollary 3. *A graph G is 3-connected if and only if G can be obtained from K_4 by iteratively splitting vertices into adjacent vertices of degree at least three.*

2 Wagner's theorem

We now finish the proof of the characterization of planar graphs by forbidden minors. We first deal with the 3-connected graphs.

Lemma 4. *If G is 3-connected and $K_5, K_{3,3} \not\preceq_m G$, then G is planar.*

Proof. We prove the claim by induction on $|V(G)|$. If $|V(G)| \leq 4$, then G is planar, and thus we can assume $|V(G)| \geq 5$. By Tutte's theorem, there exists $e = uv \in E(G)$ such that G/e is 3-connected. By symmetry, we can assume $\deg(u) \leq \deg(v)$. Since G/e is a minor of G, it contains neither K_5 nor $K_{3,3}$ as a minor. By the induction hypothesis, G/e is planar.

Consider a plane drawing of G/e. Let w be the vertex of G/e created by the contraction of e. Since G/e is 3-connected, $(G/e) - w = G - \{u, v\}$ is 2-connected, and thus every face of $(G/e) - w$ is bounded by a cycle. Let C be the cycle bounding the face of $(G/e) - w$ in which w is drawn. Since G is 3-connected, u has at least two neighbors in C. A *u-span* is a maximal subpath Q of C not containing any neighbors of u, and the *sides* of the span are the two vertices $x, y \in V(C) \setminus V(Q)$ adjacent to the ends of Q. By the maximality of Q, both x and y are adjacent to u.

Suppose first that there exists a u-span Q with sides x and y such that v has a neighbor $x' \in V(Q)$. Then every neighbor of v in C is contained in $V(Q) \cup \{x, y\}$, as otherwise if v had a neighbor $y \in V(C) \setminus (V(Q) \cup \{x, y\})$, then $C + \{ux, uy, uv, vx', vy'\}$ would be a subdivision of $K_{3,3}$, contradicting the assumption $K_{3,3} \not\preceq_m G$. Consequently, G is planar: From the drawing of G/e, delete edges wz such that z is only adjacent to v in G, thus obtaining a drawing of $G - v$. Then, draw v in the face of $G - v$ whose boundary contains Q.

Therefore, we can assume no u-span contains a neighbor of v, and thus every neighbor of v in C is also a neighbor of u. Since $\deg(u) \le \deg(v)$, it follows that u and v have exactly the same neighbors in C. Moreover, $\deg(u) = \deg(v) = \deg(w) \ge 3$, since G/e is 3-connected. But then C together with uv and three common neighbors of u and v in C forms a subdivision of K_5, contradicting the assumption that $K_5 \not\preceq_m G$. $\square$

Exercise 5. *Prove that if G is a 2-connected graph containing neither K_4 nor $K_{2,3}$ as a minor, then G is outerplanar. First, observe using the ear lemma that if $G \neq K_3$ is a 2-connected graph, then it contains an edge e such that either $G - e$ or G/e is 2-connected. Then use induction similarly to the proof of Lemma 4.*

We are now ready to finish the characterization.

Theorem 6 (Wagner)**.** *A graph G is planar if and only if $K_5, K_{3,3} \not\preceq_m G$.*

Proof. If G is planar, then all its minors are planar, and thus $K_5, K_{3,3} \not\preceq_m G$. Hence, it suffices to prove that every graph G containing neither K_5 nor $K_{3,3}$ as a minor is planar. We prove the claim by induction on $|V(G)|$.

If G is not connected, then the claim follows by the induction hypothesis applied to each component of G. If G is connected but not 2-connected, then let $G = G_1 \cup G_2$, where G_1 and G_2 are proper induced subgraphs of G intersecting in exactly one vertex v. Clearly $K_5, K_{3,3} \not\preceq_m G_1, G_2$, and thus by the induction hypothesis, G_1 and G_2 are planar. Without loss of generality, we can draw them so that v is incident with their outer faces, then glue their drawings on v to obtain a plane drawing of G.

If G is 2-connected but not 3-connected, then $G = G_1 \cup G_2$, where G_1 and G_2 are proper induced subgraphs of G intersecting in exactly two vertices u and v. Note that for $i \in \{1, 2\}$, G_i contains a path from u to v, as otherwise G would not be 2-connected. Consequently, $G_{3-i} + uv$ is a minor of G, and by the induction hypothesis, it is planar. There exist plane drawings of $G_1 + uv$ and $G_2 + uv$ such that the edge uv is incident with their outer faces. Hence, we can glue these drawings to obtain a plane drawing of $G + uv$, and thus also of G.

Finally, if G is 3-connected, then the claim follows by Lemma 4. $\square$

Exercise 7. *Prove that if a graph G contains neither K_4 nor $K_{2,3}$ as a minor, then G is outerplanar, using the result of Exercise 5 and then proceeding similarly to the proof of Theorem 6.*

Exercise 8. *Show that the result of Exercise 7 also follows from Theorem 6 using the following observation. Let G' be the graph obtained from G by adding a vertex u adjacent to all vertices of G. Since $K_4, K_{2,3} \not\preceq_m G$, we have $K_5, K_{3,3} \not\preceq_m G'$.*

3 Hadwiger's conjecture

Observation 9. *If $K_2 \not\preceq_m G$, then G has no edges, and thus $\chi(G) = 1$. If $K_3 \not\preceq_m G$, then G is a forest, and thus $\chi(G) \leq 2$.*

Let us now consider graphs not containing K_4 as a minor.

Lemma 10. *If a graph G has $n \geq 4$ vertices and at least $2n - 2$ edges, then G contains K_4 as a minor.*

Proof. We prove the claim by induction on n. For $n = 4$, the only graph with at least $2n - 2 = 6$ edges is K_4, and thus the claim holds. Hence, assume that $n \geq 5$. Without loss of generality, we can assume that G has exactly $2n - 2$ edges, as otherwise we can delete edges from G without violating the assumptions. Consequently, G has average degree $2|E(G)|/n = 4 - 4/n < 4$, and thus G contains a vertex v of degree at most three.

If $\deg(v) \leq 2$, then $G - v$ has $n - 1$ vertices and at least $(2n - 2) - 2 = 2(n - 1) - 2$ edges, and by the induction hypothesis, $K_4 \preceq_m G - v$, implying that $K_4 \preceq_m G$. Hence, we can assume that $\deg(v) = 3$. Since $K_4 \not\preceq_m G$, v has non-adjacent neighbors x and y. Then $G - v + xy \subset G/vx$ is a minor of G, and thus $K_4 \not\preceq_m G - v + xy$. Note that $|V(G - v + xy)| = n - 1$ and $|E(G - v + xy)| = (2n - 2) - 3 + 1 = 2(n - 1) - 2$. By the induction hypothesis, $K_4 \preceq_m G - v + xy \preceq_m G$. $\square$

Exercise 11. *For every $n \geq 4$, find a graph with n vertices and $2n - 3$ edges not containing K_4 as a minor. Hint: minors of outerplanar graphs are outerplanar, and K_4 is not outerplanar.*

Corollary 12. *If $K_4 \not\preceq_m G$, then G contains a vertex of degree at most three.*

Remark: Actually, any graph not containing K_4 as a minor contains a vertex of degree at most two, but this is slightly harder to prove.

Exercise 13. *Similarly prove that a graph with $n \geq 5$ vertices and at least $3n - 5$ edges contains K_5 as a minor. Hint: As in the proof of Lemma 10, we can assume the graph contains a vertex v of degree at most five, and deal by induction with the case that either* $\deg(v) = 4$, *or* $\deg(v) = 5$ *and some neigbor of v has at least to non-neighbors in $N(v)$. In case that* $\deg(v) = 5$

and each neighbor of v has at most one non-neighbor in $N(v)$, show that the graph contains K_5 as a minor by contracting a suitable edge among the vertices in $N(v)$.

More generally, the following claim (which we are not going to prove) holds.

Theorem 14. *There exists a function $f(k) = O(k\sqrt{\log k})$ such that if G has at least $f(k)|V(G)|$ edges, then G contains K_k as a minor.*

We can now bound the chromatic number of K_4-minor-free graphs.

Lemma 15. *Every graph G not containing K_4 as a minor is 3-colorable.*

Proof. We prove the claim by the induction on the number of vertices of G. Graphs with at most three vertices are 3-colorable, and thus we can assume that $|V(G)| \geq 4$. By Corollary 12, there exists $v \in V(G)$ of degree at most three. If $\deg(v) \leq 2$, then let $G' = G - v$. If $\deg(v) = 3$, then since $K_4 \not\preceq_m G$, v has non-adjacent neighbors x and y; let $G' = G/\{vx, vy\}$. In either case, $K_4 \not\preceq_m G$, and thus G' has a 3-coloring φ by the induction hypothesis. If $\deg(v) = 3$, modify φ to a 3-coloring of $G - v$ by giving both x and y the color of the vertex obtained by contracting the edges vx and vy (both x and y can have the same color, since they are non-adjacent). Thus, φ is a 3-coloring of $G - v$ such that at most two distinct colors appear among the neighbors of v. Hence, we can extend φ to a 3-coloring of G by giving v a color in $\{1, 2, 3\}$ different from these two colors. □

Exercise 16. *Similarly, using the result of Exercise 13, prove that if $K_5 \not\preceq_m G$, then G is 5-colorable.*

Hadwiger proposed the following influential hypothesis.

Conjecture 17 (Hadwiger)**.** *For every positive integer k, if $K_k \not\preceq_m G$, then $\chi(G) \leq k - 1$.*

We have shown this is true for $k \leq 4$.

Exercise 18. *Show that Hadwiger's conjecture implies the Four Color Theorem.*

Hadwiger's conjecture is known to be true for $k \leq 6$. We also know it holds approximately:

Exercise 19. *Use Theorem 14 to prove that if $K_k \not\preceq_m G$, then $\chi(G) \leq 2f(k) = O(k\sqrt{\log k})$.*

This bound was recently improved to $\chi(G) \leq O(k(\log\log k)^6)$.