Graph minors and equivalence of Wagner's and Kuratowski's theorem

Zdeněk Dvořák

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If a graph G contains a subdivision of a graph H as a subgraph, we write $H \preceq_t G$ and say that H is a *topological minor* of G. You know this notion from the statement of the well-known characterization of planar graphs.

Theorem 1 (Kuratowski)**.** *A graph G is planar if and only if $K_5, K_{3,3} \not\preceq_t G$.*

The goal of this lecture is to prove Theorem 1. To this end, it is convenient to work with a different notion of graph containment.

1 Graph minors

By *contracting an edge uv* in a graph G, we mean deleting the vertices u and v and adding a new vertex adjacent to all vertices of $(N(u) \cup N(v)) \setminus \{u, v\}$. In other words, we identify the vertices u and v to a single vertex, suppressing the loops and parallel edges that may arise. We denote by G/uv the graph obtained by this contraction.

Definition 2. *A graph H is a* minor *of G if a graph isomorphic to H can be obtained from a subgraph of G by a sequence of edge contractions. In this case, we say $H \preceq_m G$.*

Exercise 3. *If $A \subseteq G$, then A is a minor of G. If $F \preceq_m H$ and $H \preceq_m G$, then $F \preceq_m G$.*

Exercise 4. *Show that if $H \preceq_t G$, then also $H \preceq_m G$. Find a graph G such that $K_{1,4} \preceq_m G$, but $K_{1,4} \not\preceq_t G$.*

Observation 5. *If G is planar, then all minors of G are planar, and in particular $K_5, K_{3,3} \not\preceq_m G$.*

Proof. Every subgraph of a planar graph is planar, and thus it suffices to prove that contraction of an edge in a planar graph preserves planarity. Suppose we are contracing an edge $e = uv$ in a graph G drawn in the plane, and let S be the set of edges incident with u or v different from e. Consider a very small neighborhood Λ. of the drawing of e which contains only e and initial segments of the edges of S. Erase the part of the drawing of G inside Λ, draw a new vertex w inside Λ, and connect it within Λ to the points where the remainders of the edges of S intersect the boundary of Λ. This gives us a drawing of G/e in the plane. □

Exercise 6. *Show that if G is outerplanar, then all minors of G are outerplanar as well.*

Hence, we can try to characterize planar graphs in terms of their minors rather than topological minors. Indeed, the following theorem holds and we will be giving its proof shortly.

Theorem 7 (Wagner)**.** *A graph G is planar if and only if $K_5, K_{3,3} \not\preceq_m G$.*

Before we proceed with the proof of Theorem 7, let us show that it implies Kuratowski's theorem. For this, we need to consider the relationship between minors and topological minors. Let us start with a reformulation of the definition of a minor.

Definition 8. *A* model *of H in G is a function μ that*

- *to every vertex v of H assigns a connected subgraph $\mu(v)$ of G, such that for $u \neq v$, $\mu(u)$ and $\mu(v)$ are vertex disjoint, and*
- *to every edge uv of H assigns an edge $\mu(uv) = xy$ of G such that $x \in V(\mu(u))$ and $y \in V(\mu(v))$.*

Lemma 9. *A graph H is a minor of a graph G if and only if there exists a model of H in G.*

Proof. Suppose first that there exists a model μ of H in G. Delete from G all vertices and edges that do not belong to the model, i.e., are contained neither in $\bigcup_{v \in V(H)} \mu(v)$ nor in $\{\mu(e) : e \in E(H)\}$. Then, for each $v \in V(H)$, contract all edges of $\mu(v)$. Since $\mu(v)$ is connected, this shrinks it to a single vertex, and if $uv \in E(H)$, then the edge $\mu(uv)$ joins the vertex $\mu(u)$ and the vertex $\mu(v)$ in the resulting graph. Consequently, the resulting graph is isomorphic to H, and thus H is a minor of G.

Conversely, suppose H is a minor of G, and thus H is obtained from a subgraph G' of G by a sequence of contractions. For each $v \in V(H)$, let

M_v be the set of vertices of G' that end up contracted into a vertex v. Note that the graph $G'[M_v]$ is connected, as otherwise it would not be possible to contract it to a single vertex. For $e \in E(H)$, let m_e be one of the edges of G which ends up being mapped to the edge e by the contractions. Then a model μ of H in G can be defined by letting $\mu(v) = G[M_v]$ and $\mu(e) = m_e$ for each v and e. □

Exercise 10. *Note that we could add the following condition to the definition of a model μ and still conclude that presence of a minor corresponds to the existence of a model:*

- *For every $v \in V(H)$, $\mu(v)$ is a tree, and for each leaf x of the tree, there exists an edge $e \in E(H)$ such that $\mu(e)$ is incident with x.*

As we have seen in Exercise 4, the fact that H is a minor of G does not imply that H is also a topological minor of G. However, this is at least true for subcubic graphs. Given a model μ of a graph H in G and a vertex $v \in V(H)$, let $\mu^+(v)$ denote the subgraph of G obtained from $\mu(v)$ by adding all edges $\mu(e)$ such that $e \in E(H)$ is incident with v.

Lemma 11. *If H is a graph of maximum degree at most three and $H \preceq_m G$, then $H \preceq_t G$.*

Proof. Suppose $H \preceq_m G$. By Lemma 9, there exists a model μ of H in G. Without loss of generality, we can assume μ satisfies the additional condition from Exercise 10. Since H has minimum degree three, for each vertex $v \in V(H)$ one of the following claims holds.

- $\mu(v)$ is a subdivision of the star with three rays, and each leaf of $\mu(v)$ is incident with the edge $\mu(e)$ for an edge $e \in E(H)$ incident with v, or
- $\mu(v)$ is a path and both ends of the path are incident with the edge $\mu(e)$ for an edge $e \in E(H)$ incident with v, or
- $\mu(v)$ is a single vertex.

Equivalently, $\mu^+(v)$ is a subdivision of $K_{1,\deg(v)}$. Observe that it follows that the union of $\mu(v)$ for $v \in V(H)$ together with the edges $\mu(e)$ for $e \in E(H)$ is subdivision of H. Therefore, $H \preceq_t G$. □

Exercise 12. *Prove that a graph G is a forest if and only if $K_3 \not\preceq_m G$.*

We can prove something weaker for K_5.

Lemma 13. *If $K_5 \preceq_m G$, then either $K_5 \preceq_t G$, or $K_{3,3} \preceq_t G$.*

Proof. Let μ be a model of K_5 in G satisfying the additional condition from Exercise 10. If for each $v \in V(K_5)$, the graph $\mu^+(v)$ is a subdivision of $K_{1,4}$, then the union of the subgraphs and edges of the model μ gives a subdivision of K_5 in G.

Otherwise, observe that there must exist $v \in V(K_5)$ such that $\mu^+(v)$ is the subdivision of the tree T with two vertices of degree three and four leaves. Imagine we, for each $u \in V(K_5)$, contract all edges in $\mu(u)$, except that for v, we leave the middle edge of T uncontracted. This shows that G contains as a minor the graph K_5^+, obtained from K_5 by splitting one vertex into two adjacent vertices of degree three. However, $K_{3,3} \subseteq K^+5$, and thus $K_{3,3}$ is also a minor of G. $\square$

We are now ready to show that Wagner's theorem implies Kuratowski's theorem.

Proof of Theorem 1, assuming Theorem 7. Since K_5 and $K_{3,3}$ are not planar, if G is planar, then it contains neither of them as a topological minor. Cnversely, suppose that $K_5, K_{3,3} \not\preceq_t G$. By Lemmas 11 and 13, this implies that $K_5, K_{3,3} \not\preceq_m G$, and thus G is planar by Theorem 7. $\square$

Exercise 14. *Prove that Kuratowski's theorem implies Wagner's theorem.*

2 3-connectivity

The basic plan for the proof of Wagner's theorem is to proceed by induction. In a given graph G with no K_5 or $K_{3,3}$ minor, we contract an edge e. Then clearly $K_5, K_{3,3} \not\preceq_m G/e$, and thus by the induction hypothesis, G/e is planar. We can then consider a plane drawing of G/e and try to modify it to a plane drawing of G. However, there is one issue: It can be the case that the graph G/e can be drawn in many different ways in the plane, and some of them may be quite different from any possible drawing of G. To circumvent this, we consider 3-connected graphs, which have unique drawings in the following sense.

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a *homeomorphism* of the plane if f is a bijection, continuous, and the inverse f^{-1} is continuous as well. For example, rotations, reflections, and shear are homeomorphisms.

Lemma 15. *Any two plane drawings of a 3-connected graph can be transformed to each other by a homeomorphism of the plane.*

We will not actually need this Lemma, and thus we do not give its proof (you can try to think about it, though you may prefer to wait for two more lectures, as we will consider the drawings and homeomorphisms in more detail).

Exercise 16. *Show (intuitively, viewing homeomorphisms as "continuous bijective deformations of the plane", rather than arguing about them formally) this is not the case for 2-connected graphs. Hint: One of the two parts to which the graph is split on a* 2*-cut can be reflected, obtaining a non-homeomorphic drawing.*

However, considering 3-connected graphs brings another complication to the inductive argument: We need to make sure that the graph G/e is 3-connected. Fortunately, this is always possible.

Theorem 17 (Tutte)**.** *For every* 3*-connected graph* $G \neq K_4$*, there exists an edge* $e \in E(G)$ *such that* G/e *is* 3*-connected.*

We will give the proof the next time.