Number of perfect matchings

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1 Number of matchings in bipartite graphs

Consider a bipartite graph G with parts $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. The *bipartite adjacency matrix* of G is the $n \times n$ matrix C such that $C_{i,j} = 1$ if $a_i b_j \in E(G)$ and $C_{i,j} = 0$ otherwise. Note that $a_1 b_{\pi(1)}$, $a_2 b_{\pi(2)}$, ..., $a_n b_{\pi(n)}$ is a perfect matching in G if and only if $\pi(1)$, ..., $\pi(n)$ are pairwise different (i.e., π is a permutation of $\{1, \ldots, n\}$) and $C_{1,\pi(1)} = \ldots = C_{n,\pi(n)} = 1$.

The *permanent* of the $n \times n$ matrix C is

$$\operatorname{per}(C) = \sum_{\pi \text{ permutation}} \prod_{i=1}^{n} c_{i,\pi(i)}.$$

Observation 1. *If G is a bipartite graph with bipartite adjacency matrix C, then the number of perfect matchings in G is* $\operatorname{per}(C)$.

Exercise 2. *Show that if G is a bipartite graph with parts of size n and $G \neq K_{n,n}$, then G has at most $n! - (n-1)!$ perfect matchings. Find such a graph with exactly $n! - (n-1)!$ perfect matchings.*

The definition of permanent seems quite similar to the definition of the *determinant*,

$$\det(C) = \sum_{\pi \text{ permutation}} \operatorname{sgn}(\pi) \prod_{i=1}^{n} c_{i,\pi(i)},$$

where $\operatorname{sgn}(\pi) \in \{-1, 1\}$ is the *sign* of the permutation π. However, while you can determine the permanent of a matrix in polynomial time, this is likely not possible for the permanent (even the permanent of a $\{0,1\}$-matrix). The problem of determining the permanent of a $\{0,1\}$-matrix (or equivalently, the number of perfect matchings in a bipartite graph) is #P-complete, and solving it in polynomial time would imply P = NP.

Exercise 3. *Show you can determine in polynomial time whether a bipartite graph has an even or an odd number of perfect matchings. Hint: compare the definitions of the permanent and the determinant.*

Thus, in general, the best we can hope for is obtaining approximations or bounds for the number of perfect matchings.

Theorem 4 (Bregman-Cinc inequality)**.** *Let C be an $n \times n$ $\{0,1\}$-matrix, and let $r_i = \sum_{j=1}^{n} C_{i,j}$ denote the sum of the i-th row of C. Then*

$$\mathrm{per}(C) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

We will not prove this theorem. Using the upper bound $d! \leq ed(d/e)^d$, we have the following corollary (note that $\sqrt[d]{ed} \to 1$ as $d \to \infty$).

Corollary 5. *A d-regular bipartite graph with parts of size n has at most $(d!)^{n/d} \leq \left(\sqrt[d]{ed} \cdot d/e\right)^n$ perfect matchings.*

Exercise 6. *For any integer $d \geq 1$ and any integer n divisible by d, find a d-regular bipartite graph with parts of size n that has exactly $(d!)^{n/d}$ perfect matchings.*

Exercise 7. *Suppose G is a bipartite graph with both parts of size n and suppose that the average degree d of G is an integer. Show that G has at most $(d!)^{n/d}$ perfect matchings.*

For a lower bound, we use another well-known inequality, originally conjectured by Van der Waerden. A matrix is *bistochastic* if it is non-negative the sum of each row and each column is equal to 1.

Theorem 8. *If C is an $n \times n$ bistochasic matrix, then* $\mathrm{per}(C) \geq \frac{n!}{n^n}$*.*

Exercise 9. *Find an $n \times n$ bistochasic matrix such that* $\mathrm{per}(C) = \frac{n!}{n^n}$*.*

We are not going to prove Theorem 8; let us just note the following consequence.

Corollary 10. *A d-regular bipartite graph G with parts of size n has at least $(d/e)^n$ perfect matchings.*

Proof. Let C be the bipartite adjacency matrix of G. Since G is d-regular, the matrix C/d is bistochastic. Hence, using the bound $n! \geq (n/e)^n$, we have

$$\mathrm{per}(C) = d^n \mathrm{per}(C/d) \geq \tfrac{n! d^n}{n^n} \geq (d/e)^n.$$

□

Note that for non-bipartite graphs, the situation is much more complicated; the fact that 3-regular 2-edge-connected graphs have an exponential number of perfect matchings was proved only recently, and the right magnitude of the exponential is unknown.

2 Counting the matchings in planar graphs

While it is hard to count a number of matchings in a general graph, there are polynomial-time algorithms for some special graph classes. Very interestingly, there is such an algorithm for planar graphs.

First, let us introduce some more general definitions. An *even 2-factor* F on vertex set V of even size is a graph whose components are even cycles; we allow cycles of length two. Let $c(F)$ be the number of components of F, and let us define $\mathrm{sgn}(F) = (-1)^{|V|-c(F)}$. Let also $c_2(F)$ denote the number of 2-cycles of F. Note that for any two matchings M_1 and M_2 with vertex set V, the union $M_1 + M_2$ of M_1 and M_2 is an even 2-factor (with the cycles of length two corresponding to the edges belonging to both M_1 and M_2). Let $M(F)$ be the set of all pairs (M_1, M_2) of matchings such that $F = M_1 + M_2$.

Observation 11. *For any even 2-factor F, we have $|M(F)| = 2^{c(F)-c_2(F)}$.*

A permutation π is *even-cycled* if each of the cycles of π has even length. For an even 2-factor F, let $\Pi(F)$ be the set of even-cycled permutations π such that F is obtained by forgetting the orientation in the cycles of π. Note that each such permutation π satisfies $\mathrm{sgn}(\pi) = \mathrm{sgn}(F)$.

Observation 12. *For any even 2-factor F, we have $|\Pi(F)| = 2^{c(F)-c_2(F)}$.*

For distinct integers x and y, let $\lambda(x, y) = 1$ if $x < y$ and $\lambda(x, y) = -1$ if $y < x$. Let us now define a sign of a matching M with vertex set $V = \{1, \ldots, n\}$ as follows. Let σ be an arbitrary permutation of V such that the edges of M are $\sigma(1)\sigma(2)$, $\sigma(3)\sigma(4)$, $\ldots$; we let

$$\mathrm{sgn}(M) = \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{n/2} \lambda(\sigma(2i-1), \sigma(2i)).$$

Observation 13. *The sign of M does not depend on the choice of σ.*

Proof. Any two possible choices of σ can be obtained from one another by a sequence of the following operations:

- swapping $\sigma(2i-1)$ and $\sigma(2i)$ for some i

- swapping $\sigma(2i-1)$ with $\sigma(2j-1)$ and $\sigma(2i)$ with $\sigma(2j)$ for some $i \neq j$.

Recall that swapping two elements of a permutation reverses the sign of the permutation. Hence, the first operation reverses $\mathrm{sgn}(\sigma)$, but also reverses $\lambda(\sigma(2i-1), \sigma(2i))$. The second operation swaps the elements in a permutation twice, and thus it does not change $\mathrm{sgn}(\sigma)$. In either case, $\sigma(M)$ is unaffected. □

Now, let us relate the signs of matchings and even 2-factors. For an even cycle $C = v_1 v_2 \dots v_t$ whose vertices are integers, let us define $\lambda(C) = -1$ if the set D of indices i such that $v_i > v_{i+1}$ has odd size and 1 otherwise; we take the indices cyclically, i.e., by v_{t+1} we mean v_1. Note that since C is even, it does not matter in which direction we traverse C, as reversing the order replaces D by $V(C) \setminus D$. For an even 2-factor F on vertex set $\{1, \dots, n\}$, we define $\lambda(F)$ to be the product of $\lambda(C)$ over the cycles of F.

Lemma 14. *For any matchings M_1 and M_2 on the vertex set $V = \{1, \dots, n\}$, we have* $\mathrm{sgn}(M_1 + M_2) = \mathrm{sgn}(M_1)\mathrm{sgn}(M_2)\lambda(M_1 + M_2)$.

Proof. Let $\pi \in \Pi(M_1 + M_2)$ be any permutation with the same cycles as $M_1 + M_2$. Consider any cycle $C = v_1 v_2 \dots v_{2t}$ of π. Let $a(C)$ be the sequence $v_1, v_2, \dots, v_{2t}$ and $b(C)$ the sequence $v_2, v_3, \dots v_{2t}, v_1$. If the cycles of π are $C_1, \dots, C_k$, let a be the concatenation of the sequences $a(C_1), \dots, a(C_k)$ and b the concatenation of the sequences $b(C_1), \dots, b(C_k)$. Let σ_1 be the permutation mapping i to the i-th element of a, amd σ_2 the permutation mapping i to the i-th element of b. Observe that $\pi = \sigma_1^{-1} \circ \sigma_2$, and thus $\mathrm{sgn}(M_1 + M_2) = \mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$.

Moreover, by the definition we have

$$\mathrm{sgn}(M_k) = \mathrm{sgn}(\sigma_k) \cdot \prod_{i=1}^{n} \lambda(\sigma_k(2i-1), \sigma_k(2i))$$

for $k \in \{1, 2\}$, and

$$\prod_{i=1}^{n} \lambda(\sigma_1(2i-1), \sigma_1(2i))\lambda(\sigma_2(2i-1), \sigma_2(2i)) = \prod_{i=1}^{n} \lambda(i, \pi(i)) = \lambda(M_1 + M_2).$$

Combining these equalities, we obtain $\mathrm{sgn}(M_1+M_2) = \mathrm{sgn}(M_1)\mathrm{sgn}(M_2)\lambda(M_1+M_2)$. □

Let G be a graph and let $b : E(G) \to \mathbb{R}$ be an assignment of values to edges. The *Pfaffian* is defined as follows.

$$\mathrm{Pf}(G, b) = \sum_{M \text{ perfect matching of } G} \mathrm{sgn}(M) \prod_{e \in E(M)} b(e).$$

Suppose that $V(G) = \{1, \ldots, n\}$. The *antisymmetric adjacency matrix of* (G, b) is the matrix C such that $C_{u,v} = b(uv)$ if $uv \in E(G)$ and $u < v$, $C_{u,v} = -b(uv)$ if $uv \in E(G)$ and $u > v$, and $C_{u,v} = 0$ otherwise. Using the following result, we can (up to sign) compute the Pfaffian.

Lemma 15. *Let G be a graph with vertex set $\{1, \ldots, n\}$ for n even and let $b : E(G) \to \mathbb{R}$ be an assignment of values to edges. Let C be the antisymmetric adjacency matrix of (G, b). Then $\mathrm{Pf}^2(G, b) = \det(C)$.*

Proof. Consider a term $\mathrm{sgn}(\pi) \prod_{i=1}^n C_{i,\pi(i)}$ appearing in the definition of the determinant. Suppose π contains an odd cycle and π' is obtained from π by reversing this odd cycle. Clearly $\mathrm{sgn}(\pi') = \mathrm{sgn}(\pi)$, and since the cycle is odd and the matrix C is antisymmetric, we have

$$\mathrm{sgn}(\pi) \prod_{i=1}^{n} C_{i,\pi(i)} = -\mathrm{sgn}(\pi') \prod_{i=1}^{n} C_{i,\pi'(i)}.$$

Hence, these two terms cancel each other. It follows that

$$\begin{aligned}
\det(C) &= \sum_{\pi \text{ even-cycled}} \mathrm{sgn}(\pi) \prod_{i=1}^{n} C_{i,\pi(i)} \\
&= \sum_{F \subseteq G \text{ even 2-factor}} 2^{c(F)-c_2(F)} \mathrm{sgn}(F) \lambda(F) \prod_{e \in E(F)} b(e) \\
&= \sum_{M_1, M_2 \subseteq G \text{ perfect matchings}} \mathrm{sgn}(M_1)\mathrm{sgn}(M_2) \prod_{e \in E(M_1)} b(e) \prod_{e \in E(M_2)} b(e) \\
&= \left(\sum_{M \subseteq G \text{ perfect matching}} \mathrm{sgn}(M) \prod_{e \in E(M)} b(e) \right)^2 = \mathrm{Pf}^2(G, b).
\end{aligned}$$

□

A *Pfaffian function* for a graph G with vertex set $\{1, \ldots, n\}$ is a function $b : E(G) \to \{-1, 1\}$ such that for every perfect matching M of G, $\mathrm{sgn}(M) \cdot \prod_{e \in E(M)} b(e)$ is the same.

Exercise 16. *Every tree has at most one perfect matching, and thus it also has a Pfaffian function.*

Observation 17. *If there exists a Pfaffian function b for the graph G, then G has precisely $|\mathrm{Pf}(G, b)|$ perfect matchings.*

As we can determine the absolute value of the Pfaffian in polynomial time using Lemma 15, if we can find a Pfaffian function for G, then we can also determine the number of perfect matchings.

Theorem 18 (Kasteleyn)**.** *For a plane graph G with vertex set $\{1, \ldots, n\}$, a Pfaffian function can be found in polynomial time.*

Proof. Without loss of generality, we can assume G is connected. For an internal face f, let $v_1 v_2 \ldots v_t$ be the vertices encountered when traversing the boundary of f in the clockwise order, and let us define $B(f) = \{(v_1, v_2), \ldots, (v_{t-1}, v_t), (v_t, v_1)\}$. We choose the function b so that for every internal face f,

($\star$) the number of pairs $(u, v) \in B(f)$ such that $b(uv) \neq \lambda(u, v)$ is odd.

We can do this by induction: If G is a tree, then any choice of b works. Otherwise, there exists an edge e separating the outer face of G from some internal face f. We apply the induction hypothesis to obtain the restriction of b to $G - e$. Then we select $b(e) \in \{-1, 1\}$ so that ($\star$) holds for f.

Let us argue b is a Pfaffian function for G. Consider any perfect matchings M_1 and M_2 of G and the even 2-factor $F = M_1 + M_2$. Let $C = u_1 u_2 \ldots u_t$ be a cycle of F traversed in the clockwise order and let $\mathrm{Int}(C)$ denote the set of faces of G drawn inside C. Let m be the number of edges of G drawn strictly inside C. Then (with $u_{t+1} = u_1$) we have

$$\begin{aligned}\lambda(C) \prod_{e \in E(C)} b(e) &= \prod_{i=1}^{t} \lambda(u_i, u_{i+1}) b(u_i u_{i+1}) \\ &= (-1)^m \prod_{f \in \mathrm{Int}(C)} \prod_{(u,v) \in B(f)} \lambda(u, v) b(uv) \\ &= (-1)^m \prod_{f \in \mathrm{Int}(C)} (-1) = -1,\end{aligned}$$

where

- the second equality holds since for each edge uv drawn strictly inside C, the contributions $\lambda(u, v)b(uv)$ and $\lambda(v, u)b(uv)$ from the two incident faces combine to -1,
- the third equality holds by ($\star$), and
- the final one holds since the number n_C of vertices drawn inside C is even (as they are covered by the cycles of the even 2-factor F), the Euler's formula gives $n_c = (m + |C|) + 2 - (|\mathrm{Int}(C)| + 1)$, and thus m and $|\mathrm{Int}(C)|$ have the opposite parity.

Using Lemma 14, we have

$$\begin{aligned}\Big(\mathrm{sgn}(M_1)\cdot\prod_{e\in E(M_1)} b(e)\Big)\cdot\Big(\mathrm{sgn}(M_2)\cdot\prod_{e\in E(M_2)} b(e)\Big) &= \mathrm{sgn}(M_1)\mathrm{sgn}(M_2)\prod_{e\in E(F)} b(e)\\ &= \lambda(F)\mathrm{sgn}(F)\prod_{e\in E(F)} b(e)\\ &= \prod_{C \text{ cycle of } F}\Big((-1)\cdot\lambda(C)\prod_{e\in E(C)} b(e)\Big) = 1.\end{aligned}$$

Therefore, for every perfect matching M of G, $\mathrm{sgn}(M)\cdot\prod_{e\in E(M)} b(e)$ is the same, and thus b is a Pfaffian function for G. □

Exercise 19. *Choose your favourite planar graph with an even number of vertices and compute the number of its perfect matchings using the described method.*