1 Number of matchings in bipartite graphs

Consider a bipartite graph $G$ with parts $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. The bipartite adjacency matrix of $G$ is the $n \times n$ matrix $C$ such that $C_{i,j} = 1$ if $a_ib_j \in E(G)$ and $C_{i,j} = 0$ otherwise. Note that $a_1b_{\pi(1)}, a_2b_{\pi(2)}, \ldots, a_nb_{\pi(n)}$ is a perfect matching in $G$ if and only if $\pi(1), \ldots, \pi(n)$ are pairwise different (i.e., $\pi$ is a permutation of $\{1, \ldots, n\}$) and $C_{1,\pi(1)} = \ldots = C_{n,\pi(n)} = 1$.

The permanent of the $n \times n$ matrix $C$ is

$$\text{per}(C) = \sum_{\pi \text{ permutation}} \prod_{i=1}^{n} c_{i,\pi(i)}.$$

Observation 1. If $G$ is a bipartite graph with bipartite adjacency matrix $C$, then the number of perfect matchings in $G$ is $\text{per}(C)$.

Exercise 2. Show that if $G$ is a bipartite graph with parts of size $n$ and $G \neq K_{n,n}$, then $G$ has at most $n! - (n - 1)!$ perfect matchings. Find such a graph with exactly $n! - (n - 1)!$ perfect matchings.

The definition of permanent seems quite similar to the definition of the determinant,

$$\text{det}(C) = \sum_{\pi \text{ permutation}} \text{sgn}(\pi) \prod_{i=1}^{n} c_{i,\pi(i)},$$

where $\text{sgn}(\pi) \in \{-1, 1\}$ is the sign of the permutation $\pi$. However, while you can determine the permanent of a matrix in polynomial time, this is likely not possible for the permanent (even the permanent of a $\{0, 1\}$-matrix). The problem of determining the permanent of a $\{0, 1\}$-matrix (or equivalently, the number of perfect matchings in a bipartite graph) is $\#P$-complete, and solving it in polynomial time would imply $P = NP$. 

Number of perfect matchings

Zdeněk Dvořák

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Exercise 3. Show you can determine in polynomial time whether a bipartite graph has an even or an odd number of perfect matchings. Hint: compare the definitions of the permanent and the determinant.

Thus, in general, the best we can hope for is obtaining approximations or bounds for the number of perfect matchings.

Theorem 4 (Bregman-Cinc inequality). Let $C$ be an $n \times n \{0,1\}$-matrix, and let $r_i = \sum_{j=1}^{n} C_{i,j}$ denote the sum of the $i$-th row of $C$. Then

$$\text{per}(C) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$ 

We will not prove this theorem. Using the upper bound $d! \leq ed(d/e)^d$, we have the following corollary (note that $\sqrt[d]{ed} \to 1$ as $d \to \infty$).

Corollary 5. A $d$-regular bipartite graph with parts of size $n$ has at most $(d!)^{n/d}$ perfect matchings.

Exercise 6. For any integer $d \geq 1$ and any integer $n$ divisible by $d$, find a $d$-regular bipartite graph with parts of size $n$ that has exactly $(d!)^{n/d}$ perfect matchings.

Exercise 7. Suppose $G$ is a bipartite graph with both parts of size $n$ and suppose that the average degree $d$ of $G$ is an integer. Show that $G$ has at most $(d!)^{n/d}$ perfect matchings.

For a lower bound, we use another well-known inequality, originally conjectured by Van der Waerden. A matrix is bistochastic if it is non-negative the sum of each row and each column is equal to 1.

Theorem 8. If $C$ is an $n \times n$ bistochastic matrix, then

$$\text{per}(C) \geq \frac{n!}{n^n}.$$ 

Exercise 9. Find an $n \times n$ bistochastic matrix such that $\text{per}(C) = \frac{n!}{n^n}$.

We are not going to prove Theorem 8; let us just note the following consequence.

Corollary 10. A $d$-regular bipartite graph $G$ with parts of size $n$ has at least $(d/e)^n$ perfect matchings.

Proof. Let $C$ be the bipartite adjacency matrix of $G$. Since $G$ is $d$-regular, the matrix $C/d$ is bistochastic. Hence, using the bound $n! \geq (n/e)^n$, we have

$$\text{per}(C) = d^n \text{per}(C/d) \geq \frac{n!d^n}{n^n} \geq (d/e)^n.$$ 

\[ \square \]
Note that for non-bipartite graphs, the situation is much more complicated; the fact that 3-regular 2-edge-connected graphs have an exponential number of perfect matchings was proved only recently, and the right magnitude of the exponential is unknown.

2 Counting the matchings in planar graphs

While it is hard to count a number of matchings in a general graph, there are polynomial-time algorithms for some special graph classes. Very interestingly, there is such an algorithm for planar graphs.

First, let us introduce some more general definitions. An even 2-factor $F$ on vertex set $V$ of even size is a graph whose components are even cycles; we allow cycles of length two. Let $c(F)$ be the number of components of $F$, and let us define $\text{sgn}(F) = (-1)^{|V| - c(F)}$. Let also $c_2(F)$ denote the number of 2-cycles of $F$. Note that for any two matchings $M_1$ and $M_2$ with vertex set $V$, the union $M_1 + M_2$ of $M_1$ and $M_2$ is an even 2-factor (with the cycles of length two corresponding to the edges belonging to both $M_1$ and $M_2$). Let $M(F)$ be the set of all pairs $(M_1, M_2)$ of matchings such that $F = M_1 + M_2$.

**Observation 11.** For any even 2-factor $F$, we have $|M(F)| = 2^{c(F) - c_2(F)}$.

A permutation $\pi$ is even-cycled if each of the cycles of $\pi$ has even length. For an even 2-factor $F$, let $\Pi(F)$ be the set of even-cycled permutations $\pi$ such that $F$ is obtained by forgetting the orientation in the cycles of $\pi$. Note that each such permutation $\pi$ satisfies $\text{sgn}(\pi) = \text{sgn}(F)$.

**Observation 12.** For any even 2-factor $F$, we have $|\Pi(F)| = 2^{c(F) - c_2(F)}$.

For distinct integers $x$ and $y$, let $\lambda(x, y) = 1$ if $x < y$ and $\lambda(x, y) = -1$ if $y < x$. Let us now define a sign of a matching $M$ with vertex set $V = \{1, \ldots, n\}$ as follows. Let $\sigma$ be an arbitrary permutation of $V$ such that the edges of $M$ are $\sigma(1)\sigma(2)$, $\sigma(3)\sigma(4)$, $\ldots$; we let

$$\text{sgn}(M) = \text{sgn}(\sigma) \cdot \prod_{i=1}^{n/2} \lambda(\sigma(2i - 1), \sigma(2i)).$$

**Observation 13.** The sign of $M$ does not depend on the choice of $\sigma$.

**Proof.** Any two possible choices of $\sigma$ can be obtained from one another by a sequence of the following operations:

- swapping $\sigma(2i - 1)$ and $\sigma(2i)$ for some $i$
swapping $\sigma(2i - 1)$ with $\sigma(2j - 1)$ and $\sigma(2i)$ with $\sigma(2j)$ for some $i \neq j$.

Recall that swapping two elements of a permutation reverses the sign of the permutation. Hence, the first operation reverses $\text{sgn}(\sigma)$, but also reverses $\lambda(\sigma(2i-1), \sigma(2i))$. The second operation swaps the elements in a permutation twice, and thus it does not change $\text{sgn}(\sigma)$. In either case, $\sigma(M)$ is unaffected.

\[ \Box \]

Now, let us relate the signs of matchings and even 2-factors. For an even cycle $C = v_1 v_2 \ldots v_t$ whose vertices are integers, let us define $\lambda(C) = -1$ if the set $D$ of indices $i$ such that $v_i > v_{i+1}$ has odd size and 1 otherwise; we take the indices cyclically, i.e., by $v_{t+1}$ we mean $v_1$. Note that since $C$ is even, it does not matter in which direction we traverse $C$, as reversing the order replaces $D$ by $V(C) \setminus D$. For an even 2-factor $F$ on vertex set $\{1, \ldots, n\}$, we define $\lambda(F)$ to be the product of $\lambda(C)$ over the cycles of $F$.

**Lemma 14.** For any matchings $M_1$ and $M_2$ on the vertex set $V = \{1, \ldots, n\}$, we have $\text{sgn}(M_1 + M_2) = \text{sgn}(M_1)\text{sgn}(M_2)\lambda(M_1 + M_2)$.

**Proof.** Let $\pi \in \Pi(M_1 + M_2)$ be any permutation with the same cycles as $M_1 + M_2$. Consider any cycle $C = v_1 v_2 \ldots v_t$ of $\pi$. Let $a(C)$ be the sequence $v_1, v_2, \ldots, v_t$ and $b(C)$ the sequence $v_1, v_2, v_3, \ldots, v_t, v_1$. If the cycles of $\pi$ are $C_1, \ldots, C_k$, let $a$ be the concatenation of the sequences $a(C_1), \ldots, a(C_k)$ and $b$ the concatenation of the sequences $b(C_1), \ldots, b(C_k)$. Let $\sigma_1$ be the permutation mapping $i$ to the $i$-th element of $a$, and $\sigma_2$ the permutation mapping $i$ to the $i$-th element of $b$. Observe that $\pi = \sigma_1^{-1} \circ \sigma_2$, and thus $\text{sgn}(M_1 + M_2) = \text{sgn}(\pi) = \text{sgn}(\sigma_1)\text{sgn}(\sigma_2)$.

Moreover, by the definition we have

$$\text{sgn}(M_k) = \text{sgn}(\sigma_k) \cdot \prod_{i=1}^{n} \lambda(\sigma_k(2i - 1), \sigma_k(2i))$$

for $k \in \{1, 2\}$, and

$$\prod_{i=1}^{n} \lambda(\sigma_1(2i - 1), \sigma_1(2i))\lambda(\sigma_2(2i - 1), \sigma_2(2i)) = \prod_{i=1}^{n} \lambda(i, \pi(i)) = \lambda(M_1 + M_2).$$

Combining these equalities, we obtain $\text{sgn}(M_1 + M_2) = \text{sgn}(M_1)\text{sgn}(M_2)\lambda(M_1 + M_2)$. \[ \Box \]

Let $G$ be a graph and let $b : E(G) \rightarrow \mathbb{R}$ be an assignment of values to edges. The **Pfaffian** is defined as follows.

$$\text{Pf}(G, b) = \sum_{M \text{ perfect matching of } G} \text{sgn}(M) \prod_{e \in E(M)} b(e).$$

4
Suppose that $V(G) = \{1, \ldots, n\}$. The \textit{antisymmetric adjacency matrix of $(G, b)$} is the matrix $C$ such that $C_{u,v} = b(uv)$ if $uv \in E(G)$ and $u < v$, $C_{u,v} = -b(uv)$ if $uv \in E(G)$ and $u > v$, and $C_{u,v} = 0$ otherwise. Using the following result, we can (up to sign) compute the Pfaffian.

\textbf{Lemma 15.} Let $G$ be a graph with vertex set $\{1, \ldots, n\}$ for $n$ even and let $b : E(G) \to \mathbb{R}$ be an assignment of values to edges. Let $C$ be the antisymmetric adjacency matrix of $(G, b)$. Then $\text{Pf}^2(G, b) = \det(C)$.

\textit{Proof.} Consider a term $\text{sgn}(\pi) \prod_{i=1}^n C_{i,\pi(i)}$ appearing in the definition of the determinant. Suppose $\pi$ contains an odd cycle and $\pi'$ is obtained from $\pi$ by reversing this odd cycle. Clearly $\text{sgn}(\pi') = \text{sgn}(\pi)$, and since the cycle is odd and the matrix $C$ is antisymmetric, we have

\[
\text{sgn}(\pi) \prod_{i=1}^n C_{i,\pi(i)} = -\text{sgn}(\pi') \prod_{i=1}^n C_{i,\pi'(i)}.
\]

Hence, these two terms cancel each other. It follows that

\[
\det(C) = \sum_{\pi \text{ even-cycled}} \text{sgn}(\pi) \prod_{i=1}^n C_{i,\pi(i)}
\] 

\[= \sum_{F \subseteq G \text{ even } 2\text{-factor}} 2^{c_2(F)} \text{sgn}(F) \lambda(F) \prod_{e \in E(F)} b(e)
\] 

\[= \sum_{M_1, M_2 \subseteq G \text{ perfect matchings}} \text{sgn}(M_1) \text{sgn}(M_2) \prod_{e \in E(M_1)} b(e) \prod_{e \in E(M_2)} b(e)
\]

\[= \left( \sum_{M \subseteq G \text{ perfect matching}} \text{sgn}(M) \prod_{e \in E(M)} b(e) \right)^2 = \text{Pf}^2(G, b).
\]

\hfill $\Box$

A \textit{Pfaffian function} for a graph $G$ with vertex set $\{1, \ldots, n\}$ is a function $b : E(G) \to \{-1, 1\}$ such that for every perfect matching $M$ of $G$, $\text{sgn}(M) \cdot \prod_{e \in E(M)} b(e)$ is the same.

\textbf{Exercise 16.} Every tree has at most one perfect matching, and thus it also has a Pfaffian function.

\textbf{Observation 17.} If there exists a Pfaffian function $b$ for the graph $G$, then $G$ has precisely $|\text{Pf}(G, b)|$ perfect matchings.

As we can determine the absolute value of the Pfaffian in polynomial time using Lemma 15, if we can find a Pfaffian function for $G$, then we can also determine the number of perfect matchings.
Theorem 18 (Kasteleyn). For a plane graph $G$ with vertex set $\{1, \ldots, n\}$, a Pfaffian function can be found in polynomial time.

Proof. Without loss of generality, we can assume $G$ is connected. For an internal face $f$, let $v_1v_2 \ldots v_t$ be the vertices encountered when traversing the boundary of $f$ in the clockwise order, and let us define $B(f) = \{(v_1, v_2), \ldots, (v_{t-1}, v_t), (v_t, v_1)\}$. We choose the function $b$ so that for every internal face $f$,

\[(\star) \text{ the number of pairs } (u, v) \in B(f) \text{ such that } b(uv) \neq \lambda(u, v) \text{ is odd.}\]

We can do this by induction: If $G$ is a tree, then any choice of $b$ works. Otherwise, there exists an edge $e$ separating the outer face of $G$ from some internal face $f$. We apply the induction hypothesis to obtain the restriction of $b$ to $G - e$. Then we select $b(e) \in \{-1, 1\}$ so that $(\star)$ holds for $f$.

Let us argue $b$ is a Pfaffian function for $G$. Consider any perfect matchings $M_1$ and $M_2$ of $G$ and the even 2-factor $F = M_1 + M_2$. Let $C = u_1u_2 \ldots u_t$ be a cycle of $F$ traversed in the clockwise order and let $\text{Int}(C)$ denote the set of faces of $G$ drawn inside $C$. Let $m$ be the number of edges of $G$ drawn strictly inside $C$. Then (with $u_{t+1} = u_1$) we have

\[
\lambda(C) \prod_{e \in E(C)} b(e) = \prod_{i=1}^t \lambda(u_i, u_{i+1})b(u_iu_{i+1})
\]

\[
= (-1)^m \prod_{f \in \text{Int}(C)} \prod_{(u, v) \in B(f)} \lambda(u, v)b(uv)
\]

\[
= (-1)^m \prod_{f \in \text{Int}(C)} (-1) = -1,
\]

where

- the second equality holds since for each edge $uv$ drawn strictly inside $C$, the contributions $\lambda(u, v)b(uv)$ and $\lambda(v, u)b(uv)$ from the two incident faces combine to $-1$,

- the third equality holds by $(\star)$, and

- the final one holds since the number $n_C$ of vertices drawn inside $C$ is even (as they are covered by the cycles of the even 2-factor $F$), the Euler’s formula gives $n_c = (m + |C|) + 2 - (|\text{Int}(C)| + 1)$, and thus $m$ and $|\text{Int}(C)|$ have the opposite parity.
Using Lemma 14, we have

\[
\left( \text{sgn}(M_1) \cdot \prod_{e \in E(M_1)} b(e) \right) \cdot \left( \text{sgn}(M_2) \cdot \prod_{e \in E(M_2)} b(e) \right) = \text{sgn}(M_1) \text{sgn}(M_2) \prod_{e \in E(F)} b(e) \\
= \lambda(F) \text{sgn}(F) \prod_{e \in E(F)} b(e) \\
= \prod_{C \text{ cycle of } F} \left( (-1) \cdot \lambda(C) \prod_{e \in E(C)} b(e) \right) = 1.
\]

Therefore, for every perfect matching $M$ of $G$, $\text{sgn}(M) \cdot \prod_{e \in E(M)} b(e)$ is the same, and thus $b$ is a Pfaffian function for $G$. \[\square\]

**Exercise 19.** Choose your favourite planar graph with an even number of vertices and compute the number of its perfect matchings using the described method.